Algebraic Stack and its Connections with Markoff Triples

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Abstract

We introduce the Markoff equation and a group action that acts on the set of solution of the Markoff equation. We state the conjecture of Bourgain, Gamburd, and Sarnak regarding the $\mathbb{Z}/p\mathbb{Z}$ solution of Markoff equation, their progress, and the recent result by Chen that reduce the conjecture to a finite computation. Then we explain the connection of Markoff equation with character variety. Finally, we develop some algebraic geometry machinery necessary to understand Chen's work.

Contents

1 Introduction

1.1 Markoff Equation and Markoff Triples

Definition 1.1.1. The **Markoff equation** is the equation

$$x^2 + y^2 + z^2 - 3xyz = 0$$

A positive integer solution of the Markoff equation is called a **Markoff triple**.

Remark 1.1.2. The Markoff equation is symmetric in variable x, y and z, so permuting a Markoff triple gives another Markoff triple. Also, given a Markoff triple (x_0, y_0, z_0), viewing the Markoff equation as a degree 2 polynomial in variable z

$$z^2 - (3xy)z + x^2 + y^2 = 0$$

we see that $(x_0, y_0, 3x_0y_0 - z_0)$ is another Markoff triple because the solution of a quadratic equation sum to the coefficients of the linear term up to a minus sign. Let $\Gamma \subseteq \mathrm{Aut}(\mathbb{A}^3)$ be the subgroup of automorphisms of affine 3-space generated by permuting x, y, and z, and the $(x, y, z) \mapsto (x, y, 3xy - z)$. The move $(x, y, z) \mapsto (x, y, 3xy - z)$ is called the **Vieta involution**.

Then the group Γ acts on the set of Markoff triples and Markoff showed, in his 1879 paper [Mar79], that the action of Γ on the set of Markoff triple is transitive.

Instead of integer solution, we can also consider the solution of the Markoff equation in the field $\mathbb{Z}/p\mathbb{Z}$. In 2015, Bourgain, Gamburd, and Sarnak studied this question and conjectured that the action of Γ on the nontrivial $\mathbb{Z}/p\mathbb{Z}$-solution (solutions that are not $(0, 0, 0) \in \mathbb{A}^3_{\mathbb{Z}/p\mathbb{Z}}$) is also transitive. To support their claim, Bourgain, Gamburd, and Sarnak established their conjecture for all but a sparse, but infinite, set of primes. Moreover, for sufficiently large prime, there always exist a large orbit.

Definition 1.1.3. Let p be a prime and $\Gamma \subseteq \mathrm{Aut}(\mathbb{A}^3_{\mathbb{Z}/p\mathbb{Z}}) = \mathrm{Aut}((\mathbb{Z}/p\mathbb{Z})[x, y, z])$ be the subgroup generated by permuting x, y, z and $(x, y, z) \mapsto (x, y, 3xy - z)$. Let

$$X(p) = \{(x, y, z) \in \mathbb{A}^3_{\mathbb{Z}/p\mathbb{Z}} \mid x^2 + y^2 + z^2 - 3xyz = 0\}$$

and $X^*(p) = X(p) \setminus \{(0, 0, 0)\}$. Let $\mathbb{E}_{\text{bgs}} = \{p \text{ prime} \mid \Gamma \curvearrowright X^*(p) \text{ is not transitive}\}$ be the set of exceptional prime.

Conjecture 1.1.4. [BGS15] For a prime p, Γ acts on $X(p)$ with two orbits: $(0, 0, 0)$ and $X^*(p)$.

Theorem 1.1.5. [BGS15] For all $\varepsilon > 0$, $|\{p \in \mathbb{E}_{\text{bgs}} \mid p \leq x\}| = O(x^{\varepsilon})$.

Theorem 1.1.6. [BGS15] For all $\varepsilon > 0$ and there exist sufficiently large $N \in \mathbb{N}$ such that $p \geq N$ implies, there exist an orbit $C(p)$ of $\Gamma \curvearrowright X^*(p)$ such that

$$|X^*(p) \setminus C(p)| \leq p^{\varepsilon}$$

Remark 1.1.7. In 2020, Chen proved a divisibility theorem on the cardinalities of Nielsen equivalence classes of generating pairs of finite groups. As a corollary, p divides the size of every Γ-orbit in $X^*(p)$. Combined with the result of Bourgain, Gamburd, and Sarnak, this establishes conjecture 1.1.4 for all but finitely many primes, reducing the conjecture to a finite computation.

Theorem 1.1.8. [Che21] For all prime p, p divided every Γ-orbit of $X^*(p)$.

1.2 Representation Variety and Character Variety

Definition 1.2.1. Let Γ be a group. Define $A(\Gamma, \mathrm{SL}_2)$, called the **universal representation algebra**, by

$$A(\Gamma, \mathrm{SL}_2) = \mathbb{Z}[a_{ij}(\gamma) \text{ for } \gamma \in \Gamma \text{ and } i, j \in \{1, 2\}]/I$$

where I is the ideal

$$I = \left\langle \begin{array}{c} a_{ij}(e) - \delta_{ij}, a_{ij}(\gamma\delta) - \sum_k a_{ik}(\gamma)a_{kj}(\delta), \\ \det(\sigma(\gamma)) - 1 \text{ for } \gamma \in \Gamma \text{ and } i, j \in \{1, 2\} \text{ and } \gamma, \delta \in \Gamma \end{array} \right\rangle$$

The map $\sigma : \Gamma \to \mathrm{SL}_2(A(\Gamma, \mathrm{SL}_2))$ defined by $\gamma \mapsto \sigma(\gamma)$ is a representation of Γ is called the **universal representation** of Γ in SL_2. The SL_2-**representation variety** of Γ is the affine scheme $\mathrm{Spec}(A(\Gamma, \mathrm{SL}_2))$.

Definition 1.2.2. Let Γ be a group. Define $R(\Gamma, \mathrm{SL}_2)$, called the **universal character ring**, by

$$R(\Gamma, \mathrm{SL}_2) = \frac{\mathbb{Z}[t_\gamma, \gamma \in \Gamma]}{\langle t_e - 2, t_\gamma t_\delta - t_{\gamma\delta} - t_{\gamma^{-1}\delta} \rangle}$$

The SL_2-**character variety** of Γ is the affine scheme $\mathrm{Spec}(R(\Gamma, \mathrm{SL}_2))$.

Remark 1.2.3. Since the representation variety and character variety are affine schemes, we can view them as functors from the category of schemes to the category of sets, or functor from the category of rings to the category of sets.

Remark 1.2.4. The relation $s(\gamma)s(\delta) - s(\gamma\delta) - s(\gamma^{-1}\delta)$ in the denominator of the definition of $R(\Gamma, \mathrm{SL}_2)$ (definition 1.2.2) is known as the **Fricke identity** which is satisfied by the trace of two matrices in SL_2. Suppose $M, N \in \mathrm{SL}_2(R)$ for some ring R, by Cayley

Hamilton

$$\begin{aligned} M^2 - \operatorname{tr}(M)M + \det(M)I_2 = 0 \\ MN + \det(M)M^{-1}N = \operatorname{tr}(M)N \\ \operatorname{tr}(MN) + \det(M)\operatorname{tr}\big(M^{-1}N\big) = \operatorname{tr}(M)\operatorname{tr}(N) \\ \operatorname{tr}(M)\operatorname{tr}(N) - \operatorname{tr}(MN) - \operatorname{tr}\big(M^{-1}N\big) = 0 \end{aligned}$$

Theorem 1.2.5. (Fricke) Let $\Pi = \langle a, b \rangle$ be a free group generated by two elements. Then

$$\frac{\mathbb{Z}[t_\gamma, \gamma \in \Pi]}{t_e - 2, t_\gamma t_\delta - t_{\gamma\delta} - t_{\gamma^{-1}\delta}} \simeq \mathbb{Z}[t_a, t_b, t_{ab}]$$

This result is due to Fricke [Fri96], a proof can be found in [Gol05] and [Che21, Section 6.2]. In particular, it shows $\mathrm{Ch}(\Pi, \mathrm{SL}_2) \simeq \mathrm{Spec}(\mathbb{Z}[t_a, t_b, t_{ab}]) \simeq \mathbb{A}^3$.

1.3 Orientable Surface and Mapping Class Group

Definition 1.3.1. A **surface** is a 2-dimensional manifold. A **closed surface** is a compact surface with no boundary.

Theorem 1.3.2. (Classification of closed surfaces) Any connected closed surface is homeomorphic to a surface in one of the following 3 families:

(1) the 2-sphere S^2,

(2) the orientable surface of genus g, for some $g \geq 1$, denoted Σ_g,

(3) the non-orientable surface of genus h, for some $h \geq 1$, denoted N_h.

Remark 1.3.3. From the classification of closed surfaces, we can obtain a classification of orientable compact surfaces with boundary. Given an orientable closed surface, removing finitely many open discs will give an orientable compact surface with boundary. Conversely, given an orientable compact surface with boundary, the boundary is a 1-dimensional manifold, which must be a disjoint union of circles, implying it comes from an orientable closed surface, with some open discs removed. Moreover, the precise location of the removed disc does not matter because the classification is up to homeomorphism.

Therefore, an orientable compact surface with boundary is determined, up to homeomorphism, by two invariants, the number of genus, and the number of discs removed. We call surface with n discs removed n-**puntured**. We denote the n-punctured orientable surface of genus g by $\Sigma_{g,n}$

Here, we are interested in the one-punctured torus $\Sigma_{1,1}$. $\Sigma_{1,1}$ can be visualized as follows

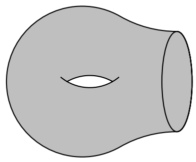


Figure 1.

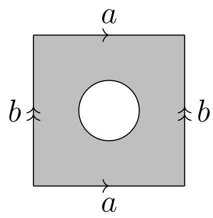


Figure 2.

Denote the fundamental group of $\Sigma_{1,1}$ by $\Pi_{1,1}$. We see that $\Pi_{1,1}$ is isomorphic to the free group generated by a, b by deformation retracting Figure 2 to the four edges of the square, and applying van Kampen's theorem. The class of the boundary curve in $\Pi_{1,1}$ is the commutator $[a, b] = aba^{-1}b^{-1}$. By theorem 1.2.5, we see that $\mathrm{Ch}(\Pi_{1,1}, \mathrm{SL}_2) \simeq \mathrm{Spec}(\mathbb{Z}[t_a, t_b, t_{ab}]) \simeq \mathbb{A}^3$.

From now on, we write $\mathrm{Ch}_{1,1}$ for $\mathrm{Ch}(\Pi_{1,1}, \mathrm{SL}_2)$.

Remark 1.3.4. Let $\Pi = \langle a, b\rangle$ be a free group generated by two elements and R be a ring. Suppose $\varphi : \Pi \to \mathrm{SL}_2(R)$ be a group homomorphism. Write

$$\begin{aligned} x &= \operatorname{tr}\varphi(a) \\ y &= \operatorname{tr}\varphi(b) \\ z &= \operatorname{tr}\varphi(ab) \end{aligned}$$

By Cayley Hamilton, for $M \in \mathrm{GL}_2(R)$

$$\begin{aligned} M^2 - \operatorname{tr}(M)M + \det(M)I_2 &= 0 \\ M + \det(M)M^{-1} &= \operatorname{tr}(M)I_2 \\ \operatorname{tr}(M) + \det(M)\operatorname{tr}\big(M^{-1}\big) &= 2\operatorname{tr}(M) \\ \operatorname{tr}(M) &= \det(M)\operatorname{tr}\big(M^{-1}\big) \end{aligned}$$

$\varphi(a), \varphi(b), \varphi(ab) \in \mathrm{SL}_2(R)$ implies $\det\varphi(a) = \det\varphi(b) = \det\varphi(ab) = 1$ which implies

$$\begin{aligned} x &= \operatorname{tr}\varphi(a) = \operatorname{tr}\varphi(a^{-1}) \\ y &= \operatorname{tr}\varphi(b) = \operatorname{tr}\varphi(b^{-1}) \\ z &= \operatorname{tr}\varphi(ab) = \operatorname{tr}\varphi(b^{-1}a^{-1}) \end{aligned}$$

By Cayley Hamilton again, for $M, N \in \mathrm{GL}_2(R)$

$$\begin{aligned} M^2 - \operatorname{tr}(M)M + \det(M)I_2 &= 0 \\ MN + \det(M)N^{-1}M &= \operatorname{tr}(M)N \\ \operatorname{tr}(MN) + \det(M)\operatorname{tr}\big(N^{-1}N\big) &= \operatorname{tr}(M)\operatorname{tr}(N) \end{aligned}$$

It follows that

$$\begin{aligned}
\operatorname{tr}\varphi(a^2)+2 &= \operatorname{tr}\varphi(a)^2 = x^2\\
\operatorname{tr}\varphi(b^2)+2 &= \operatorname{tr}\varphi(b)^2 = y^2\\
\operatorname{tr}\varphi(a^{-1}b) &= \operatorname{tr}\varphi(a^{-1})\operatorname{tr}\varphi(b) - \operatorname{tr}\varphi(ab) = xy - z\\
\operatorname{tr}\varphi(aba^{-1}b) &= \operatorname{tr}\varphi(ab)\operatorname{tr}\varphi(a^{-1}b) - \operatorname{tr}\bigl(abb^{-1}a\bigr)\\
&= z(xy-z) - x^2 + 2 = xyz - z^2 - x^2 + 2\\
\operatorname{tr}\varphi(aba^{-1}b^{-1}) &= \operatorname{tr}\varphi(aba^{-1})\operatorname{tr}\varphi(b^{-1}) - \operatorname{tr}\bigl(aba^{-1}b\bigr)\\
&= y^2 - (xyz - z^2 - x^2 + 2) = x^2 + y^2 + z^2 - xyz - 2
\end{aligned}$$

So in $\mathrm{Ch}_{1,1}$, $t_{[a,b]} = t_{aba^{-1}b^{-1}} = t_a^2 + t_b^2 + t_{ab}^2 - t_a t_b t_{ab} - 2$.

Definition 1.3.5. Let S be an orientable surface with boundary ∂S. The **mapping class group** of S, denoted by $\mathrm{MCG}(S)$ is the group of orientation preserving automorphism of S fixing ∂S, modulo the equivalence relation given by homotopy.

Remark 1.3.6. Let (S,x) be an orientable surface with boundary. An element of the mapping class group $\mathrm{MCG}(S)$ is an equivalence class of orientation preserving automorphism that fixes the boundary. Each automorphism $\varphi \in \mathrm{Aut}(S)$ induces an automorphism of the fundamental group $\pi_1(S,x)$ after fixing a path from x to $\varphi(x)$ Homotopic automorphism of the surface induces the same automorphism on the fundamental group. Therefore, an element $\mathrm{MCG}(S)$ induces an automorphism of the fundamental group since every representative induce the same automorphism of the fundamental group. Thus, we get an action of the mapping class group on the fundamental group $\mathrm{MCG}(S) \curvearrowright \pi_1(S,x)$. Then the mapping class group act on the character variety $\mathrm{MCG}(S) \curvearrowright \mathrm{Ch}(\pi_1(S,x), \mathrm{SL}_2)$ by permuting the variable.

Remark 1.3.7. Let S be an orientable surface with boundary. Examples of automorphism of S that fix the boundary of S are **Dehn twists**. Let γ be a simple closed curve in S. A Dehn twist around γ is defined to be removing a small tabular neighborhood of γ, viewing it as an annulus, twisting one end by 2π radian and fixing the other end glue back to S.

Theorem 1.3.8. ([FM12, Theorem 4.9, Theorem 4.13, Theorem 4.14]) $\mathrm{MCG}(\Sigma_{g,n})$ is generated by Dehn twists.

Remark 1.3.9. By considering explicit generator of $\mathrm{MCG}(\Sigma_{1,1})$ and consider its action on $\mathrm{Ch}_{1,1}(\mathbb{Z}/p\mathbb{Z}) \simeq (\mathbb{Z}/p\mathbb{Z})^3$ are composition of permuting the variables and Vieta involution. Therefore, $\mathrm{MCG}(\Sigma_{1,1})$ can be viewed as a subgroup of Γ as in remark 1.1.2. Moreover, the class of the boundary curve in the fundamental group is conjugated after applying an automorphism of the surface that fix the boundary. So $t_{[a,b]} = t_a^2 + t_b^2 + t_{ab}^2 - t_a t_b t_{ab} - 2$ in $\mathrm{Ch}_{1,1}$ is invariant under the action of $\mathrm{MCG}(\Sigma_{1,1})$.

1.4 Connections with Markoff Triples

Definition 1.4.1. Define the **modified character variety**, denoted by $\mathrm{Ch}_{1,1;k}$, by defining $\mathrm{Ch}_{1,1;-2}(R) \subseteq \mathrm{Ch}_{1,1}(R) \simeq \mathbb{A}^3_R$ to be the set of points $(x, y, z) \in \mathbb{A}^3_R$ such that $x^2 + y^2 + z^2 - xyz - 2 = k$. Define the **relative modified character variety**, denoted by $\mathrm{Ch}^\bullet_{1,1;-2}(R)$, by $\mathrm{Ch}_{1,1;-2} \setminus \{(0,0)\}$.

Remark 1.4.2. Then $\mathrm{Ch}_{1,1;-2}(\mathbb{Z}/p\mathbb{Z})$ is the set of points $(x, y, z) \in (\mathbb{Z}/p\mathbb{Z})^3$ such that $x^2 + y^2 + z^2 - xyz = 0$. Then the action of $\mathrm{MCG}(\Sigma_{1,1})$ on $\mathrm{Ch}_{1,1}(\mathbb{Z}/p\mathbb{Z})$ factors through to an action of $\mathrm{Ch}_{1,1;-2}(\mathbb{Z}/p\mathbb{Z})$ because $t_{[a,b]}$ is invariant under the action of $\mathrm{MCG}(\Sigma_{1,1})$. We also get an action $\mathrm{MCG}(\Sigma_{1,1}) \curvearrowright \mathrm{Ch}^\bullet_{1,1;-2}(\mathbb{Z}/p\mathbb{Z})$.

Remark 1.4.3. There exists a bijection between the integer solution of

$$x^2 + y^2 + z^2 - 3xyz = 0 \quad \text{and} \quad x^2 + y^2 + z^2 - xyz = 0$$

given by $(x_0, y_0, z_0) \mapsto (3x_0, 3y_0, 3z_0)$. This bijection continue to hold in $\mathbb{Z}/p\mathbb{Z}$ for $p \neq 3$. Under this bijection, we get an isomorphism $\mathrm{Ch}^\bullet_{1,1;-2}(\mathbb{Z}/p\mathbb{Z}) \simeq X^*(p)$. So to show the action of Γ on $X^*(p)$ is transitive (definition 1.1.3), it suffices to show $\mathrm{MCG}(\Sigma_{1,1}) \curvearrowright \mathrm{Ch}^\bullet_{1,1;-2}(\mathbb{Z}/p\mathbb{Z})$ is transitive.

$$\begin{array}{ccc} \mathrm{MCG}(\Sigma_{1,1}) & \curvearrowright & \mathrm{Ch}^\bullet_{1,1;-2}(\mathbb{Z}/p\mathbb{Z}) \\ \downarrow & & \wr\| \\ \Gamma & \curvearrowright & X^*(p) \end{array}$$

Theorem 1.4.4. [Che21] For all prime p, p divided every Γ-orbit of $X^*(p)$.

Remark 1.4.5. Chen's method involves heavy algebraic geometry machinery like algebraic space and algebraic stack. Motivated by the above, I use the Stacks Project as the main reference to study algebraic space and algebraic stack in the rest of this exposition.

2 Stack and Stack in Groupoids

2.1 Fibered Category

Definition 2.1.1. [Sta24, 00VH] A **site** is a category $\mathcal{C}$ with a set $\mathrm{Cov}(\mathcal{C})$, where an element of $\mathrm{Cov}(\mathcal{C})$ is a family of morphisms in $\mathcal{C}$ with fixed target $\{U_i \to U\}_{i\in I}$, called **coverings** of C, satisfying the following conditions

(1) If $V \to U$ is an isomorphism, then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.

(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each $i \in I$ we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.

(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$, then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

Definition 2.1.2. Let $\mathcal{C}$ be a category, $p : \mathcal{S} \to \mathcal{C}$ be a functor (We say $\mathcal{S}$ is a **category over** $\mathcal{C}$), and $U \in \mathrm{Ob}(\mathcal{C})$ be an object in $\mathcal{C}$.

(1) [Sta24, 02XH] The **fiber category** over U, denoted by $\mathcal{S}_U$, is the category with objects

$$\mathrm{Ob}(\mathcal{S}_U) = \{x \in \mathrm{Ob}(\mathcal{S}) \mid p(x) = U\}$$

and morphisms

$$\mathrm{Mor}_{\mathcal{S}_U}(x, y) = \{\varphi \in \mathrm{Mor}_{\mathcal{S}}(x, y) \mid p(\varphi) = \mathrm{id}_U\}$$

(2) [Sta24, 001G] The **category of object over** U, denoted by $\mathcal{C}/U$, is the category with objects

$$\mathrm{Ob}(\mathcal{C}/U) = \{V \to U \text{ morphism in } \mathcal{C} \text{ with target } U\}$$

and morphisms

$$\mathrm{Mor}_{\mathcal{C}/U}(V \xrightarrow{\varphi} U, W \xrightarrow{\psi} U) = \{\chi \in \mathrm{Mor}_{\mathcal{C}}(V, W) \mid \psi \circ \chi = \varphi\}$$

(3) [Sta24, 00Z0] Assume that $\mathcal{C}$ is also a site. The **localization of the site** $\mathcal{C}$ at the object U is the site $\mathcal{C}/U$ where a family of morphism $\{V_i \to V\}_{i\in I}$ of objects over U is a covering of $\mathcal{C}/U$ if and only if it is a covering in $\mathcal{C}$.

Definition 2.1.3. [Sta24, 02XM] Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. $\mathcal{S}$ is a **fibered category** over $\mathcal{C}$ if for every $U \in \mathrm{Ob}(\mathcal{C})$, every $x \in \mathrm{Ob}(\mathcal{S}_U)$, and every $f : V \to U$ morphism in $\mathcal{C}$ with target U, there exist lift $f^*x \to x$ of f satisfying the following universal property: for every $z \in \mathrm{Ob}(\mathcal{S})$ with morphisms $\varphi : z \to x$ and $g : p(z) \to V$ such that $p(\varphi) = f \circ g$, there exist a unique lift $z \to f^*x$ of $p(z) \to V$ such that the following diagram commute

$$\begin{tikzcd}
z \arrow[dr, dashed, "\exists!"'] \arrow[drr, bend left, "\varphi"] \arrow[dd, squiggly, "p"] & & \\
& f^*x \arrow[r] \arrow[d, squiggly, "p"] & x \arrow[d, squiggly, "p"] \\
p(z) \arrow[r, "g"'] & V \arrow[r, "f"'] & U
\end{tikzcd}$$

Remark 2.1.4. f^*x in definition 2.1.3 can be thought of as the "fiber product of V and x over U" or "base change of x via $V \to U$". By a standard argument f^*x is unique up to unique isomorphism: suppose y, z both satisfy the universal property defining $f * x$, then the unique lifts $y \to z$ and $z \to y$ of id_V are inverses of each other

$$\begin{tikzcd}
z \arrow[dr, dashed, "\exists!"] \arrow[ddrr, bend right, "\mathrm{id}"'] \arrow[ddrrr, bend left] \arrow[ddd, squiggly] & & & \\
& y \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[dd, squiggly] & & \\
& & z \arrow[r] \arrow[d, squiggly] & x \arrow[d, squiggly] \\
V \arrow[r, "\mathrm{id}"'] & V \arrow[r, "\mathrm{id}"'] & V \arrow[r, "f"'] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
y \arrow[dr, dashed, "\exists!"] \arrow[ddrr, bend right, "\mathrm{id}"'] \arrow[ddrrr, bend left] \arrow[ddd, squiggly] & & & \\
& z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[dd, squiggly] & & \\
& & y \arrow[r] \arrow[d, squiggly] & x \arrow[d, squiggly] \\
V \arrow[r, "\mathrm{id}"'] & V \arrow[r, "\mathrm{id}"'] & V \arrow[r, "f"'] & U
\end{tikzcd}$$

Example 2.1.5. [Sta24, 02XV] Let $\mathcal{C}$ be a category, and $F : \mathcal{C}^{\mathrm{opp}} \to \mathsf{Categories}$ be a contravariant functor. For a morphism $f : U \to V$, we write f^* for the morphism (covariant functor) $F(f) : F(V) \to F(U)$. We construct a fibered category $\mathcal{S}_F$ over $\mathcal{C}$ as follows.

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}$$

and for $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \varphi) \mid f \in \mathrm{Mor}_{\mathcal{C}}(V, U), \varphi \in \mathrm{Mor}_{F(V)}(y, f^*x)\} \\
&= \coprod_{f \in \mathrm{Mor}_{\mathcal{C}}(U,V)} \mathrm{Mor}_{F(V)}(y, f^*x)
\end{aligned}$$

which is well-defined because $F(f) : F(U) \to F(V)$ and $f^*x = F(f)(x) \in F(V)$. Suppose $(U, x), (V, y), (W, z) \in \mathrm{Ob}(\mathcal{S}_F)$ with morphisms $(f, \varphi) : (V, y) \to (U, x)$ and $(g, \psi) : (W, z) \to (V, y)$, define the composition by $(f, \varphi) \circ (g, \psi) = (f \circ g, g^*\varphi \circ \psi)$

$$\begin{tikzcd}
(W, z) \arrow[r, "(g,\psi)"] \arrow[rr, bend right, "(f\circ g, g^*\varphi\circ\psi)"'] & (V, y) \arrow[r, "(f,\varphi)"] & (U, x)
\end{tikzcd}$$

which is well-defined because $\psi : z \to g^*y$, $\varphi : y \to f^*x$, and $g^*\varphi : g^*y \to g^*f^*x = (f \circ g)^*x$ ($g^* : F(V) \to F(W)$ is a covariant functor). The identity morphism for an object $(U, x) \in \mathrm{Ob}(\mathcal{S}_F)$ is $(\mathrm{id}_U, \mathrm{id}_x)$ and associativity of composition holds. So $\mathcal{S}_F$ is a category.

Define covariant functor $p_F : \mathcal{S}_F \to \mathcal{C}$ by $p_F(U, x) = U$ and $p_F(f, \varphi) = f$. To verify $\mathcal{S}_F$ is a fibered category over $\mathcal{C}$, it suffices to show given $f : U \to V$, and $(U, x) \in \mathrm{Ob}(\mathcal{S}_F)$, there exist an object in $(\mathcal{S}_F)_V$ satisfying the universal property in definition 2.1.3.

We show that $(V, f^*x) \in \mathrm{Ob}(\mathcal{S}_F)$ together with the map $(f, \mathrm{id}_{f^*x}) : (V, f^*x) \to (U, x)$ satisfies the universal property. Let $(W, z) \in \mathrm{Ob}(\mathcal{S}_F)$ be an object with morphisms $g : W \to V$ and $(h, \psi) : (W, z) \to (U, x)$ such that $p_F(h, \psi) = f \circ g$

$$\begin{tikzcd}
(W,z) \arrow[drr, bend left, "(h,\psi)"] \arrow[dd, squiggly, "p_F"'] & & \\
& (V,f^*x) \arrow[r, "(f,\mathrm{id}_{f^*x})"'] \arrow[d, squiggly, "p_F"] & (U,x) \arrow[d, squiggly, "p_F"] \\
W \arrow[r, "g"'] & V \arrow[r, "f"'] & U
\end{tikzcd}$$

Then $h = p_F(h, \psi) = f \circ g$. $(g, \psi) : (W, z) \to (V, f^*x)$ make the diagram commute because $p_F(g, \psi) = g$ and by definition of composition

$$(f, \mathrm{id}_{f^*x}) \circ (g, \psi) = (f \circ g, g^* \mathrm{id}_{f^*x} \circ \psi) = (f \circ g, \mathrm{id}_{g^*f^*x} \circ \psi) = (f \circ g, \psi)$$

where the second equality hold because g^* is a functor

$$\begin{tikzcd}
(W,z) \arrow[drr, bend left, "(h,\psi)=(f\circ g,\psi)"] \arrow[dr, "(g,\psi)"'] \arrow[dd, squiggly, "p_F"'] & & \\
& (V,f^*x) \arrow[r, "(f,\mathrm{id}_{f^*x})"'] \arrow[d, squiggly, "p_F"] & (U,x) \arrow[d, squiggly, "p_F"] \\
W \arrow[r, "g"'] & V \arrow[r, "f"'] & U
\end{tikzcd}$$

Suppose $(g', \psi') : (W, z) \to (V, f^*x)$ also make the diagram commute, then $p_F(g', \psi) = g$ implies $g = g'$. By definition of composition in $\mathcal{S}_F$, the diagram commute implies $\psi = \psi'$ because

$$(f \circ g, \psi) = (f, \mathrm{id}_{f^*x}) \circ (g', \psi') = (f, \mathrm{id}_{f^*x}) \circ (g, \psi') = (f \circ g, g^* \mathrm{id}_{f^*x} \circ \psi') = (f \circ g, \psi')$$

So $\mathcal{S}_F$ is a fibered category over $\mathcal{C}$.

2.2 Stack

The goal of this section is to define a stack over a site.

Definition 2.2.1. [Sta24, 02XN] Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibered category. A **choice of pullbacks** for $p : \mathcal{S} \to \mathcal{C}$ is given by a choice of morphism $f^*x \to x$ lying over f satisfying the universal property in definition 2.1.3 for any morphism $f : V \to U$ of $\mathcal{C}$ and any $x \in \mathrm{Ob}(\mathcal{S}_U)$.

Definition 2.2.2. Let $\mathcal{C}$ be a category.

(1) [Sta24, 02X6] A **presheaf** on $\mathcal{C}$ is a contravariant functor F from $\mathcal{C}$ to Sets, the

category of sets. A morphism of presheaves is a natural transformation. Denote the category of presheaves on $\mathcal{C}$ by $\mathsf{PSh}(\mathcal{C})$.

(2) [Sta24, 00V8] A presheaf F is said to be a **subpresheaf** of another presheaf G if for every $U \in \mathrm{Ob}(\mathcal{C})$, $F(U) \subseteq G(U)$ and for every morphism $\varphi : V \to U$ in $\mathcal{C}$, $F(\varphi) = G(\varphi)|_{F(U)}$.

Definition 2.2.3. [Sta24, 00VM] Let $\mathcal{C}$ be a site, and let F be a presheaf on $\mathcal{C}$. Let $\{U_i \to U\}_{i\in I}$ be an element of $\mathrm{Cov}(\mathcal{C})$. By condition (3) in definition 2.1.1, all fiber product $U_i \times_U U_j$ exist in $\mathcal{C}$. Then we have the following maps

$$\begin{tikzcd} & U_i \times_U U_j \arrow[dl, "\mathrm{pr}_i^{(i,j)}"'] \arrow[dr, "\mathrm{pr}_j^{(i,j)}"] & \\ U_i \arrow[dr] & & U_j \arrow[dl] \\ & U & \end{tikzcd} \qquad \begin{tikzcd} & F(U_i \times_U U_j) & \\ F(U_i) \arrow[ur, "F\left(\mathrm{pr}_i^{(i,j)}\right)"] & & F(U_j) \arrow[ul, "F\left(\mathrm{pr}_j^{(i,j)}\right)"'] \\ & F(U) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

For each $i \in I$, define the following two functions

$$\begin{aligned} \mathrm{pr}_0^* : \prod_{i\in I} F(U_i) &\to \prod_{(j,k)\in I^2} F(U_j \times_U U_k) \\ (s_i)_{i\in I} &\mapsto \left(F\left(\mathrm{pr}_j^{(j,k)}\right)(s_j)\right)_{(j,k)\in I^2} \end{aligned} \qquad \begin{aligned} \mathrm{pr}_1^* : \prod_{i\in I} F(U_i) &\to \prod_{(j,k)\in I^2} F(U_j \times_U U_k) \\ (s_i)_{i\in I} &\mapsto \left(F\left(\mathrm{pr}_k^{(j,k)}\right)(s_k)\right)_{(j,k)\in I^2} \end{aligned}$$

F is a **sheaf** if for every covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$, the diagram

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod_{(j,k)\in I^2} F(U_j \times_U U_k)$$

represents the first arrow as the equalizer of pr_0^* and pr_1^*. Or equivalently, the image of $F(U)$ in $\prod_{i\in I} F(U_i)$ is equal to

$$\left\{ (s_i)_{i\in I} \in \prod_{i\in I} F(U_i) \;\middle|\; \mathrm{pr}_0^*((s_i)_{i\in I}) = \mathrm{pr}_1^*((s_i)_{i\in I}) \right\}$$

Definition 2.2.4. [Sta24, 02ZB] Let $p : \mathcal{S} \to \mathcal{C}$ be a fibered category over a category $\mathcal{C}$. Given $U \in \mathrm{Ob}\,\mathcal{C}$ and $x, y \in \mathrm{Ob}(\mathcal{S}_U)$, the **presheaf of morphisms** from x to y is the presheaf $\mathrm{Mor}(x, y)$ on $\mathcal{C}/U$ defined by

$$(f : V \to U) \longmapsto \mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y)$$

(The lemma below shows this is in fact a presheaf) The **presheaf of isomorphisms** from x to y is the subpresheaf $\mathrm{Isom}(x, y)$ of the presheaf $\mathrm{Mor}(x, y)$ on $\mathcal{C}/U$ defined by

$$(f : V \to U) \longmapsto \mathrm{Isom}_{\mathcal{S}_V}(f^*x, f^*y)$$

Lemma 2.2.5. [Sta24, 026A] Let $p : \mathcal{S} \to \mathcal{C}$ be a fibered category over a category $\mathcal{C}$. Given $U \in \mathrm{Ob}\,\mathcal{C}$ and $x, y \in \mathrm{Ob}(\mathcal{S}_U)$, then $\mathrm{Mor}(x, y)$ is a presheaf on $\mathcal{C}/U$

Proof. To simplify notation, denote $\mathrm{Mor}(x,y)$ by $M_{x,y}$. It suffices to show $M_{x,y}$ is a contravariant functor from $\mathcal{C}/U$ to Sets. Given a morphism $g : V_1 \to V_2$ in $\mathcal{C}/U$ where $f_1 : V_1 \to U$ and $f_2 : V_2 \to U$, define $M_{x,y}(g) : \mathrm{Mor}_{\mathcal{S}_{V_2}}(f_2^*x, f_2^*y) \to \mathrm{Mor}_{\mathcal{S}_{V_1}}(f_1^*x, f_1^*y)$ as follows: given $\varphi \in \mathrm{Mor}_{\mathcal{S}_{V_2}(f_2^*x, f_2^*y)}$ let $M_{x,y}(g)(\varphi) : f_1^*x \to f_1^*y$ be the unique morphism induced by the universal property of f_1^*y and the morphisms id_{V_1} and $f_1^*x \xrightarrow{f^*g} f_2^*x \xrightarrow{\varphi} f_2^*y \longrightarrow y$

$$\begin{tikzcd}
f_1^*x \arrow[r, "f_2^*g"] \arrow[d, dashed] \arrow[rr, bend right] & f_2^*x \arrow[r] \arrow[d, "\varphi"] & x \\
f_1^*y \arrow[r, "f_2^*g"] \arrow[rr, bend right] & f_2^*y \arrow[r] & y
\end{tikzcd}$$

$$\begin{tikzcd}
V_1 \arrow[r, "g"] \arrow[rr, bend right, "f_1"'] & V_2 \arrow[r, "f_2"] & U
\end{tikzcd}$$

Applying in universal properties, we see if $(f : V \to U) \in \mathrm{Ob}(\mathcal{C}/U)$, then $M_{x,y}(\mathrm{id}_V) = \mathrm{id}_{M_{x,y}(V)}$, and if $g_1 : V_1 \to V_2$ and $g_2 : V_2 \to V_3$ are morphisms in $\mathcal{C}/U$ where $f_i : V_i \to U$ for $i = 1, 2, 3$, then $M_{x,y}(g_2 \circ g_1) = M_{x,y}(g_2) \circ M_{x,y}(g_1)$. $\square$

Definition 2.2.6. [Sta24, 026B] Let $p : \mathcal{S} \to \mathcal{C}$ be a fibered category over a category $\mathcal{C}$. Make a choice of pullbacks. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$ be a family of morphisms of $\mathcal{C}$. Assume all fiber products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ exists.

(1) A **descent datum** (X_i, φ_{ij}) in $\mathcal{S}$ relative to the family $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$ is given by an object X_i of $\mathcal{S}_{U_i}$ for each $i \in I$, an isomorphism $\varphi_{ij} : \mathrm{pr}_0^* X_i \to \mathrm{pr}_1^* X_j$ in $\mathcal{S}_{U_i\times_U U_j}$ in for each pair $(i,j) \in I^2$ such that for every triple $(i,j,k) \in I^3$, the following diagram in the category $\mathcal{S}_{U_i\times_U U_j \times_U U_k}$ commutes

$$\begin{tikzcd}
\mathrm{pr}_0^* X_i \arrow[rr, "\mathrm{pr}_{02*}\varphi_{ik}"] \arrow[dr, "\mathrm{pr}_{01}^*\varphi_{ij}"'] & & \mathrm{pr}_2^* X_k \\
& \mathrm{pr}_1^* X_j \arrow[ur, "\mathrm{pr}_{02}^*\varphi_{jk}"'] &
\end{tikzcd}$$

(2) A **morphism of descent datum** $\psi : (X_i, \varphi_{ij}) \to (X_i', \varphi_{ij}')$ is given by a family $\psi = (\psi_i)_{i\in I}$ of morphism $\psi_i : X_i \to X_i'$ in $\mathcal{S}_{U_i}$ such that for every pair $(i,j) \in I^2$, the following diagram in the category $\mathcal{S}_{U_i\times_U U_j}$ commutes

$$\begin{tikzcd}
\mathrm{pr}_0^* X_i \arrow[r, "\varphi_{ij}"] \arrow[d, "\mathrm{pr}_0^*\psi_i"'] & \mathrm{pr}_1^* X_j \arrow[d, "\mathrm{pr}_0^*\psi_j"] \\
\mathrm{pr}_0^* X_i' \arrow[r, "\varphi_{ij}'"'] & \mathrm{pr}_1^* X_j
\end{tikzcd}$$

(3) The **category of descent data** relative to $\mathcal{U}$ is denoted by $\mathsf{DD}(\mathcal{U})$.

Lemma 2.2.7. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibered category and $\{f_i : U_i \to U\}_{i\in I}$ a family of morphisms. Assume all fiber products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ exists. Let

$X \in \mathrm{Ob}(\mathcal{S}_U)$. Then $(f_i^* X, (f_i \times f_j)^* \mathrm{id}_X)$ is a descent datum in $\mathcal{S}$ relative to the family $\{f_i : U_i \to U\}_{i \in I}$.

Proof. By definition of fibered category, $f_i^* X \in \mathrm{Ob}(\mathcal{S}_U)$. For each pair $(i,j) \in I^2$, the following diagram commute

$$\begin{tikzcd} & U_i \arrow[dr, "f_i"] & \\ U_i \times_U U_j \arrow[ur, "\mathrm{pr}_0"] \arrow[rr, "f_i \times f_j"] \arrow[dr, "\mathrm{pr}_1"'] & & U \\ & U_j \arrow[ur, "f_j"'] & \end{tikzcd}$$

So $\mathrm{pr}_0^* f_i^* X \simeq (f_i \circ \mathrm{pr}_0)^* X \simeq (f_i \times f_j)^* X \simeq (f_j \circ \mathrm{pr}_1)^* X \simeq \mathrm{pr}_1^* f_j^* X$ where isomorphisms here are unique in the sense of remark 2.1.4. Then $(f_i \times f_j)^* \mathrm{id}_X$ defines an isomorphism $\mathrm{pr}_0^* f_i^* X \to \mathrm{pr}_1^* f_j^* X$. For each triple $(i,j,k) \in I^3$, the following diagram commute

$$\begin{tikzcd} \mathrm{pr}_0^* f_i^* X \arrow[rr, "\mathrm{pr}_{02}^*(f_i \times f_k)^* \mathrm{id}_X"] \arrow[dr, "\mathrm{pr}_{01}^*(f_i \times f_j)^* \mathrm{id}_X"'] & & \mathrm{pr}_2^* f_k^* X \\ & \mathrm{pr}_1^* f_j^* X \arrow[ur, "\mathrm{pr}_{12}^*(f_j \times f_k)^* \mathrm{id}_X"'] & \end{tikzcd}$$

because each object is isomorphic to $(f_i \times f_j \times f_k)^* X$. So $(f_i^* X, (f_i \times f_j)^* \mathrm{id}_X)$ is a descent datum relative to the family $\{f_i : U_i \to U\}_{i \in I}$. $\square$

Definition 2.2.8. [Sta24, 026E] Let $p : \mathcal{S} \to \mathcal{C}$ be a fibered category over a category $\mathscr{C}$. Make a choice of pullbacks. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$. Assume all fiber products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ exists.

(1) Given an object X of $\mathcal{S}_U$, the **trivial descent datum** is the descent datum (X, id_X) relative to the family $\{\mathrm{id}_U : U \to U\}$.

(2) Given an object X of $\mathcal{S}_U$, the **canonical descent datum** relative to $\{f_i : U_i \to U\}_{i \in I}$ is the $(f_i^* X, (f_i \times f_j)^* \mathrm{id}_X)$. This descent datum is denoted by $(f_i^* X, can)$.

(3) A descent datum (X_i, φ_{ij}) relative to $\{f_i : U_i \to U\}_{i \in I}$ is **effective** if there exist $X \in \mathrm{Ob}(\mathcal{S}_U)$ such that (X_i, φ_{ij}) is isomorphic to $(f_i^* X, can)$ in the category $\mathsf{DD}(\mathcal{U})$ of descent datum.

Definition 2.2.9. [Sta24, 026F] Let $\mathcal{C}$ be a site. A **stack** over $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ satisfying the following conditions

(1) $p : \mathcal{S} \to \mathcal{C}$ is a fibered category,

(2) for any $U \in \mathrm{Ob}(\mathcal{C})$ and for any $x, y \in \mathcal{S}_U$, the presheaf $\mathrm{Mor}(x, y)$ is a sheaf on the site $\mathcal{C}/U$, and

(3) for any covering $\mathcal{U} = \{U_i \to U\}$ in $\mathcal{C}$, all descent datum in $\mathcal{S}$ relative to $\mathcal{U}$ is effective.

2.3 Stack in Groupoids

The goal of this section is to define a stack in groupoids over a site.

Definition 2.3.1. [Sta24, 0018] A **groupoid** is a category where every morphism is an isomorphism.

Definition 2.3.2. [Sta24, 003S] Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. $\mathcal{S}$ is **fibered in groupoids** over $\mathcal{C}$ if

(1) For every morphism $f : V \to U$ in $\mathcal{C}$ and every $x \in \mathcal{S}_U$, there exist a lift $\varphi : y \to x$ of f:

$$\begin{tikzcd}
y \arrow[r, dashed, "\exists\varphi"] \arrow[d, squiggly, "p"] & x \arrow[d, squiggly, "p"] \\
V \arrow[r, "f"'] & U
\end{tikzcd}$$

(2) For every pair of morphism $\varphi : y \to x$, $\psi : z \to x$ and every morphism $f : p(z) \to p(y)$ such that $p(\varphi) \circ f = p(\psi)$, there exist a unique lift $\chi : z \to y$ of f such that $\varphi \circ \chi = \psi$:

$$\begin{tikzcd}
y \arrow[dr, "\varphi"] \arrow[dd, squiggly] & & z \arrow[ll, dashed, "\exists!\chi"'] \arrow[dl, "\psi"'] \arrow[dd, squiggly] \\
& x \arrow[dd, squiggly] & \\
p(y) \arrow[dr, "p(\varphi)"'] & & p(z) \arrow[ll, "f"'] \arrow[dl, "p(\psi)"] \\
& p(x) &
\end{tikzcd}$$

Lemma 2.3.3. [Sta24, 003V] Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. The following are equivalent

(1) $\mathcal{S}$ is fibered in groupoids over $\mathcal{C}$.

(2) $\mathcal{S}$ is a fibered category over $\mathcal{C}$ and for each $U \in \mathrm{Ob}(\mathcal{C})$, the fiber category $\mathcal{S}_U$ is a groupoid.

Proof. (1) $\Longrightarrow$ (2) Suppose $\mathcal{S}$ is fibered in groupoids over $\mathcal{C}$.

($\mathcal{S}$ is a fibered category over $\mathcal{C}$): Let $U \in \mathrm{Ob}(\mathcal{C})$, $x \in \mathcal{S}_U$, and $f : V \to U$ be a morphism in $\mathcal{C}$. Then the map $\varphi : y \to x$ given by condition (1) in definition 2.3.2 is the desired morphism in definition 2.1.3 because given $z \in \mathrm{Ob}(\mathcal{S})$ with morphism $z \to x$ and $g : p(z) \to V$, the unique lift of g given by condition (2) in definition 2.3.2 in groupoids is a morphism $z \to y$ making the following diagram commute

$$\begin{tikzcd}
z \arrow[rrd, bend left] \arrow[rd, dashed] \arrow[dd, squiggly, "p"] & & \\
 & y \arrow[r] \arrow[d, squiggly, "p"] & x \arrow[d, squiggly, "p"] \\
p(z) \arrow[r, "g"'] & V \arrow[r, "f"'] & U
\end{tikzcd}$$

(Each fiber category is a groupoid): Let $U \in \mathrm{Ob}(\mathcal{C})$, $x, y \in \mathrm{Ob}(\mathcal{S}_U)$, and $f \in \mathrm{Mor}_{\mathcal{S}_U}(x, y)$. It suffices to show f is an isomorphism, i.e., f have a two-sided inverse. Condition (2) in definition 2.3.2 induces unique morphisms $g \in \mathrm{Mor}_{\mathcal{S}_U}(y, x)$ and $h \in \mathrm{Mor}_{\mathcal{S}_U}(x, y)$ as follows

$$\begin{tikzcd}
x \arrow[dr, "f"'] \arrow[dd, squiggly] & & y \arrow[ll, dashed, "\exists! g"'] \arrow[dl, "\mathrm{id}"] \arrow[dd, squiggly] \\
 & y \arrow[dd, squiggly] & \\
U \arrow[dr, "\mathrm{id}"'] & & U \arrow[ll, "\mathrm{id}"'] \arrow[dl, "\mathrm{id}"] \\
 & U &
\end{tikzcd}
\qquad
\begin{tikzcd}
y \arrow[dr, "g"'] \arrow[dd, squiggly] & & x \arrow[ll, dashed, "\exists! h"'] \arrow[dl, "\mathrm{id}"] \arrow[dd, squiggly] \\
 & x \arrow[dd, squiggly] & \\
U \arrow[dr, "\mathrm{id}"'] & & U \arrow[ll, "\mathrm{id}"'] \arrow[dl, "\mathrm{id}"] \\
 & U &
\end{tikzcd}$$

Then $f \circ g = \mathrm{id}_y$ and $g \circ h = \mathrm{id}_x$. g is in fact a two-sided inverse of f because $f = h$:

$$f = f \circ \mathrm{id}_x = f \circ (g \circ h) = (f \circ g) \circ h = \mathrm{id}_y \circ h = h$$

$(2) \implies (1)$ Suppose $\mathcal{S}$ is a fibered category over $\mathcal{C}$ and each fiber category is a groupoid. Condition 1 in the definition of fibered in groupoids is automatically satisfied by $\mathcal{S}$ being a fibered category over $\mathcal{C}$. To verify condition 2 in definition 2.3.2, suppose the following commutative diagram is given

$$\begin{tikzcd}
y \arrow[dr, "\varphi"'] \arrow[dd, squiggly] & & z \arrow[dl, "\psi"] \arrow[dd, squiggly] \\
 & x \arrow[dd, squiggly] & \\
p(y) \arrow[dr, "p(\varphi)"'] & & p(z) \arrow[ll, "f"'] \arrow[dl, "p(\psi)"] \\
 & p(x) &
\end{tikzcd}$$

it suffices to show $\exists! z \to y$ making the entire diagram commute. Since $\mathcal{S}$ is a fibered category over $\mathcal{C}$, there exists $p(\varphi)^* y \in \mathcal{S}_{p(y)}$ with a lift of $p(\varphi)$, unique lift $i : y \to p(\varphi)^* y$

of $\mathrm{id}_{p(y)}$ and unique lift $j : z \to p(\varphi)^*y$ of f such that the following diagram commute

$$\begin{tikzcd}
 & & p(\varphi)^*x \arrow[ddr] \arrow[dd, squiggly] & & \\
y \arrow[urr, dashed, "\exists! i"] \arrow[drrr, "\varphi"] \arrow[dd, squiggly] & & & & z \arrow[ull, dashed, "\exists! j"'] \arrow[dl, "\psi"] \arrow[dd, squiggly] \\
 & & p(y) \arrow[ddr, "p(\varphi)"] \arrow[dll, equal] & x \arrow[dd, squiggly] & \\
p(y) \arrow[drrr, "p(\varphi)"'] & & & & p(z) \arrow[ull, "f"'] \arrow[dl, "p(\psi)"] \\
 & & & p(x) &
\end{tikzcd}$$

i is a morphism in $\mathcal{S}_{p(y)}$ which is a groupoid by assumption. Then i^{-1} exist and $i^{-1} \circ j : z \to y$ is a lift of f. Uniqueness of $i^{-1} \circ j$ follows from remark 2.1.4. $\square$

Example 2.3.4. [Sta24, 0049] Let $\mathcal{C}$ be a category, and $F : \mathcal{C}^{\mathrm{opp}} \to \mathsf{Groupoids}$ be a contravariant functor. For a morphism $f : U \to V$, we write f^* for the morphism (covariant functor) $F(f) : F(V) \to F(U)$. We construct a category $\mathcal{S}_F$ fibered in groupoids over $\mathcal{C}$ as follows.

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}$$

and for $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \varphi) \mid f \in \mathrm{Mor}_{\mathcal{C}}(V, U), \varphi \in \mathrm{Mor}_{F(V)}(y, f^*x)\} \\
&= \coprod_{f \in \mathrm{Mor}_{\mathcal{C}}(U,V)} \mathrm{Mor}_{F(V)}(y, f^*x)
\end{aligned}$$

which is well-defined because $F(f) : F(U) \to F(V)$ and $f^*x = F(f)(x) \in F(V)$. Suppose $(U, x), (V, y), (W, z) \in \mathrm{Ob}(\mathcal{S}_F)$ with morphisms $(f, \varphi) : (V, y) \to (U, x)$ and $(g, \psi) : (W, z) \to (V, y)$, define the composition by $(f, \varphi) \circ (g, \psi) = (f \circ g, g^*\varphi \circ \psi)$

$$\begin{tikzcd}
(W, z) \arrow[r, "{(g,\psi)}"] \arrow[rr, bend right, "{(f\circ g, g^*\varphi\circ\psi)}"'] & (V, y) \arrow[r, "{(f,\varphi)}"] & (U, x)
\end{tikzcd}$$

which is well-defined because $\psi : z \to g^*y$, $\varphi : y \to f^*x$, and $g^*\varphi : g^*y \to g^*f^*x = (f \circ g)^*x$ ($g^* : F(V) \to F(W)$ is a covariant functor). The identity morphism for an object $(U, x) \in \mathrm{Ob}(\mathcal{S}_F)$ is $(\mathrm{id}_U, \mathrm{id}_x)$ and associativity of composition holds. So $\mathcal{S}_F$ is a category.

Define covariant functor $p_F : \mathcal{S}_F \to \mathcal{C}$ by $p_F(U, x) = U$ and $p_F(f, \varphi) = f$. To verify $\mathcal{S}_F$ is a category fibered in groupoids over $\mathcal{C}$, it suffices to show it is a fibered category over $\mathcal{C}$ and each fiber category is a groupoid by lemma 2.3.3. By example 2.1.5, $\mathcal{S}_F$ is a fibered category over $\mathcal{C}$. So it remains to show for each fiber category is a groupoid. Let $U \in \mathrm{Ob}(\mathcal{C})$, by definition of the functor F

$$\mathrm{Ob}((\mathcal{S}_F)_U) = \{(U, x) \mid x \in F(U)\}$$

and $\mathrm{Mor}_{(\mathcal{S}_F)_U}((U,y),(U,x)) = \{(\mathrm{id}_U, \varphi) \mid \varphi \in \mathrm{Mor}_{F(U)}(y,x)\}$. Since $F(U)$ is a groupoid, it follows that the inverse of a morphism (id_U, φ) in $\mathrm{Mor}_{(\mathcal{S}_F)_U}$ is $(\mathrm{id}_U, \varphi^{-1})$. Then $(\mathcal{S}_F)_U$ is a groupoid because every morphism in $(\mathcal{S}_F)_U$ is an isomorphism.

Definition 2.3.5. [Sta24, 02Y0] A **discrete category** is a category where the only morphisms are the identity morphisms.

Definition 2.3.6. [Sta24, 0043] Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. $\mathcal{S}$ is **fibered in sets** over $\mathcal{C}$ if $\mathcal{S}$ is fibered in groupoids over $\mathcal{C}$ and all fiber category over $\mathcal{C}$ are discrete.

Remark 2.3.7. A discrete category is a groupoid because identity morphism is isomorphism. The data of a discrete category is no more than its collection of objects. So we may view a set as a discrete category, and therefore a groupoid.

Example 2.3.8. [Sta24, 0049] Let $\mathcal{C}$ be a category, and $F : \mathcal{C}^{\mathrm{opp}} \to \mathsf{Sets}$ be a contravariant functor. For a morphism $f : U \to V$, we write f^* for the morphism (covariant functor) $F(f) : F(V) \to F(U)$. We construct a category $\mathcal{S}_F$ fibered in sets over $\mathcal{C}$ as follows.

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U,x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in F(U)\}$$

and for $(U,x),(V,y) \in \mathrm{Ob}(\mathcal{S}_F)$

$$\mathrm{Mor}_{\mathcal{S}_F}((V,y),(U,x)) = \{f \in \mathrm{Mor}_{\mathcal{C}}(V,U) \mid f^*x = y\}$$

The identity morphism for an object $(U,x) \in \mathrm{Ob}(\mathcal{S}_F)$ is id_U and associativity of composition holds. So $\mathcal{S}_F$ is a category.

Define covariant functor $p_F : \mathcal{S}_F \to \mathcal{C}$ by $p_F(U,x) = U$ and $p_F(f) = f$. Viewing $F(U)$ as a groupoid as in remark 2.3.7. $\mathcal{S}_F$ is a fibered in groupoid by example 2.3.4. Then $\mathcal{S}_F$ is fibered in sets because each fiber category is $F(U)$, which is a set, and therefore discrete.

Remark 2.3.9. The category $\mathcal{S}_F$ in example 2.3.8 is known as the **category of elements** of the functor $F : \mathcal{C} \to \mathsf{Sets}$. It will be used to define what it means for a category over the category of schemes to be representable by an algebraic space (definition 4.4.2)

Definition 2.3.10. [Sta24, 02ZI] A **stack in groupoids** over a site $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ such that

(1) $p : \mathcal{S} \to \mathcal{C}$ is fibered in groupoids over $\mathcal{C}$.

(2) For every $U \in \mathrm{Ob}(\mathcal{C})$ and every $x, y \in \mathrm{Ob}(\mathcal{S}_U)$, the presheaf $\mathrm{Isom}(x,y)$ is a sheaf on the site $\mathcal{C}/U$.

(3) For every covering $\mathcal{U} = \{U_i \to U\}$ in $\mathcal{C}$, all descent data (x_i, φ_{ij}) for $\mathcal{U}$ are effective.

3 Smooth and Étale Maps

3.1 Homological Algebra

Definition 3.1.1. A category $\mathcal{A}$ is a **preadditive** category if each morphism set $\mathrm{Mor}_{\mathcal{A}}(x, y)$ is endowed with the structure of an abelian group such that the compositions

$$\mathrm{Mor}(x, y) \times \mathrm{Mor}(y, z) \longrightarrow \mathrm{Mor}(x, z)$$

are bilinear, i.e., if $f_1, f_2 \in \mathrm{Mor}(x, y)$, $g_1, g_2 \in \mathrm{Mor}(y, z)$, then

$$(g_1 + g_2) \circ (f_1 + f_2) = g_1 \circ f_1 + g_1 \circ f_2 + g_2 \circ f_1 + g_2 \circ f_2$$

Sometimes a preadditive category is also called an **ab-enriched category** or a **ringoid**. A functor $F : \mathcal{A} \to \mathcal{B}$ between two preadditive category is **additive** if for all $x, y \in \mathrm{Ob}(\mathcal{A})$, $F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$ is an abelian group homomorphism.

Joke 3.1.2.

- A monoid can be viewed as a one object category.
- A group can be viewed as a one object category such that every morphism is an isomorphism.
- A ring can be viewed as a one object preadditive category.
- A preadditive category with potentially more than one object is called a ringoid.
- A category where every morphism is an isomorphism with potentially more than one object is called a groupoid.

	One object	Multiple objects
No condition on morphisms	Monoid	?
Every morphism is an isomorphism	Group	Groupoid
Morphisms form an abelian group	Ring	Ringoid

So a category with potentially more than one object and no restriction on morphisms should be called a monoidoid. But that is just a category!

Definition 3.1.3. Let $\mathcal{A}$ be a preadditive category and $f : x \to y$ be a morphism.

(1) A **kernel** of f is a morphism $i : z \to x$ such that $f \circ i = 0$ and for any $i' : z' \to x$ such that $f \circ i' = 0$, there exists a unique morphism $g : z' \to z$ such that $i' = i \circ g$

$$\begin{tikzcd} z \arrow[r, "i"] & x \arrow[r, "f"] & y \\ z' \arrow[u, dashed, "\exists! g"] \arrow[ur, "i'"'] & & \end{tikzcd}$$

When a kernel of f exists, it is denoted by $\ker f \to x$.

(2) A **cokernel** of f is a morphism $p : y \to z$ such that $p \circ f = 0$ and for any $p' : y \to z'$ such that $p' \circ f = 0$, there exist a unique morphism $g : z \to z'$ such that $g' = g \circ p$

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[r, "p"] \arrow[dr, "p'"'] & z \arrow[d, dashed, "\exists! g"] \\ & & z' \end{tikzcd}$$

When a cokernel of f exist, it is denoted by $y \to \operatorname{coker} f$.

(3) If a kernel of f exist, then a **coimage** of f is a cokernel of the morphism $\ker f \to x$. When a kernel and coimage exist, it is denoted by $x \to \operatorname{coim} f$.

(4) If a cokernel of f exist, then an **image** of f is a kernel of the morphism $y \to \operatorname{coker} f$. When a cokernel and image exist, it is denoted by $\operatorname{im} f \to y$.

Lemma 3.1.4. [Sta24, 0E43] Let $\mathcal{A}$ be a preadditive category and $f : x \to y$ be a morphism.

(1) If a kernel of f exists, then $i : \ker f \to x$ is a monomorphism.

(2) If a cokernel of f exists, then $p : y \to \operatorname{coker} f$ is an epimorphism.

(3) If a kernel and a coimage of f exist, then $x \to \operatorname{coim} f$ is an epimorphism.

(4) If a cokernel and an image of f exist, then $\operatorname{im} f \to x$ is a monomorphism.

Proof.

(1) Suppose $g, h : z \to \ker f$ are two morphisms such that $i \circ g = i \circ h$, it suffices to show $g = h$. $\mathcal{A}$ is a preadditive category implies $i \circ (g - h) = i \circ g - i \circ h = 0$. Then $f \circ i \circ (g - h) = 0$ which means by the universal property of kernel, there exist a unique morphism $z \to \ker f$ such that the following diagram commute

$$\begin{tikzcd} \ker f \arrow[r, "i"] & x \arrow[r, "f"] & y \\ z \arrow[u, dashed, "\exists!"] \arrow[ur, "i\circ(g-h)=0"'] & & \end{tikzcd}$$

Now, both $0 : z \to \ker f$ and $(g - h) : z \to \ker f$ make the diagram commute. By uniqueness, $0 = g - h$ which implies $g = h$ as desired.

(2) Suppose $g, h : \operatorname{coker} f \to z$ are two morphisms such that $g \circ p = h \circ p$, it suffices to show $g = h$. $\mathcal{A}$ is a preadditive category implies $(g - h) \circ p = g \circ p - h \circ p = 0$. Then $(g - h) \circ p \circ f = 0$ which means by the universal property of cokernel, there exist a unique

morphism $\operatorname{coker} f \to z$ such that the following diagram commute

$$\begin{tikzcd}
x \arrow[r, "f"] & y \arrow[r, "p"] \arrow[dr, "(g-h)\circ p=0"'] & \operatorname{coker} f \arrow[d, dashed, "\exists!"] \\
& & z
\end{tikzcd}$$

Now, both $0 : \operatorname{coker} f \to z$ and $(g-h) : \operatorname{coker} f \to z$ make the diagram commute. By uniqueness, $0 = g - h$ which implies $g = h$ as desired.

(3) This follows from (2) because $x \to \operatorname{coim} f$ is the cokernel of $\ker f \to x$.

(4) This follows from (1) because $\operatorname{im} f \to y$ is the kernel of $y \to \operatorname{coker} f$. □

Lemma 3.1.5. [Sta24, 0107] Let $f : x \to y$ be a morphism in a preadditive category such that kernel, cokernel, image, coimage all exist. Then f can be factored uniquely as $x \to \operatorname{coim} f \to \operatorname{im} f \to y$.

Proof. Name the morphisms as labeled in the diagram below

$$\begin{tikzcd}
\ker f \arrow[r, "i_x"] & x \arrow[r, "f"] \arrow[d, "p_x"] & y \arrow[r, "p_y"] & \operatorname{coker} f \\
& \operatorname{coim} f & \operatorname{im} f \arrow[u, "i_y"] &
\end{tikzcd}$$

$p_y \circ f = 0$ by definition of cokernel, which means there exist unique morphism $\varphi : x \to \operatorname{im} f$ such that the diagram commute because $i_y : \operatorname{im} f \to y$ is the kernel of $p_y : y \to \operatorname{coker} f$. Then $i_y \circ \varphi \circ i_x = f \circ i_x = 0$ by that commutativity of the diagram and the definition of kernel. i_y is a monomorphism by lemma 3.1.4, which implies $\varphi \circ i_x = 0$. Then there exist unique morphism $\psi : \operatorname{coim} f \to \operatorname{im} f$ such that the diagram commute because $p_x : x \to \operatorname{coim} f$ is the cokernel of $i_x : \ker f \to x$

$$\begin{tikzcd}
\ker f \arrow[r, "i_x"] & x \arrow[r, "f"] \arrow[d, "p_x"] \arrow[dr, dashed, "\varphi"] & y \arrow[r, "p_y"] & \operatorname{coker} f \\
& \operatorname{coim} f \arrow[r, dashed, "\psi"'] & \operatorname{im} f \arrow[u, "i_y"] &
\end{tikzcd}$$

□

Definition 3.1.6. Let $\mathcal{A}$ be a category

(1) $\mathcal{A}$ is an **additive** category if it is preadditive and finite products exist.

(2) $\mathcal{A}$ is a **preabelian** category if it is additive and every morphism have a kernel and a cokernel.

(3) $\mathcal{A}$ is an **abelian** category if it is preabelian and for every morphism f, the natural map $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism.

Definition 3.1.7. A **chain complex** $A_\bullet$ in an preadditive category $\mathcal{A}$ is a collection of object $\{A_i \in \operatorname{Ob}(\mathcal{A}) \mid i \in \mathbb{Z}\}$ and a collection of morphism $\{d_i : A_i \to A_{i-1} \mid i \in \mathbb{Z}\}$ such that $d_{i-1} \circ d_i = 0$ for all $i \in \mathbb{Z}$

$$\cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots$$

A morphism of chain complexes $f : A_\bullet \to B_\bullet$ is a family of morphisms $\{f_i : A_i \to B_i\}$ such that for all $i \in \mathbb{Z}$, the following diagram commute

$$\begin{tikzcd} A_i \arrow[r, "d_i"] \arrow[d, "f_i"'] & A_{i-1} \arrow[d, "f_{i-1}"] \\ B_i \arrow[r, "d_i"'] & B_{i-1} \end{tikzcd}$$

The category of chain complexes of $\mathcal{A}$ is denoted by $\mathsf{Ch}(\mathcal{A})$. If $\mathcal{A}$ is a additive category, the full subcategory consisting of objects of the form

$$\cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

is denoted by $\mathsf{Ch}_{\geq 0}(\mathcal{A})$.

Remark 3.1.8. Any additive category $\mathcal{A}$ can be identified with the full subcategory of $\mathsf{Ch}(\mathcal{A})$ consisting of chain complexes that are zero except in degree 0 by the functor

$$\begin{aligned} \mathcal{A} &\longrightarrow \mathsf{Ch}(\mathcal{A}) \\ A &\longmapsto (\cdots \to 0 \to A \to 0 \to \cdots) \end{aligned}$$

Definition 3.1.9. Let $A_\bullet$ be a chain complex in an abelian category $\mathcal{A}$. For all $i \in \mathbb{Z}$, the i-th **homology group** of $A_\bullet$ is defined by

$$H_i(A_\bullet) = \ker(d_i)/\operatorname{im}(d_{i+1})$$

(the cokernel of $\operatorname{im}(d_{i+1}) \to \ker(d_i)$). If $f : A_\bullet \to B_\bullet$ is a morphism of chain complexes in $\mathcal{A}$, then we get an induced morphism $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$ because kernel of d_i get maps to kernel of d_i and image of d_{i+1} get maps to image of d_{i+1}. Therefore, $H_i : \mathsf{Ch}(\mathcal{A}) \to \mathcal{A}$ is a functor.

Definition 3.1.10. Let $\mathcal{A}$ be a preadditive category.

(1) A **homotopy** h between a pair of morphisms of chain complex $f, g : A_\bullet \to B_\bullet$ is a collection of morphisms $h_i : A_i \to B_{i+1}$ such that $f_i - g_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$ for all $i \in \mathbb{Z}$

$$\begin{tikzcd} \cdots \arrow[r] & A_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, shift right, "f_{i+1}"'] \arrow[d, shift left, "g_{i+1}"] & A_i \arrow[r, "d_i"] \arrow[dl, "h_i"'] \arrow[d, shift right, "f_i"'] \arrow[d, shift left, "g_i"] & A_{i-1} \arrow[r] \arrow[dl, "h_{i-1}"'] \arrow[d, shift right, "f_{i-1}"'] \arrow[d, shift left, "g_{i-1}"] & \cdots \\ \cdots \arrow[r] & B_{i+1} \arrow[r, "d_{i+1}"'] & B_i \arrow[r, "d_i"'] & B_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

Two morphisms $f, g : A_\bullet \to B_\bullet$ is **homotopic** if there exist a homotopy between f and g.

(2) A morphism $f : A_\bullet \to B_\bullet$ is a **homotopy equivalence** if there exist a morphism $g : B_\bullet \to A_\bullet$ such that there exist a homotopy between $b \circ a$ and $\mathrm{id}_{A_\bullet}$ and there

exist a homotopy between $a \circ b$ and $\mathrm{id}_{B_\bullet}$. $A_\bullet$ and $B_\bullet$ is **homotopy equivalent** if there exist a morphism between them that is a homotopy equivalence.

Definition 3.1.11. Let $\mathcal{A}$ be an abelian category. A morphism $f : A_\bullet \to B_\bullet$ is a **quasi-isomorphism** if the induced map $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$ is an isomorphism for all $i \in \mathbb{Z}$. $A_\bullet$ is **quasi-isomorphic** to $B_\bullet$ if there exist a morphism from $A_\bullet$ to $B_\bullet$ that is a quasi-isomorphism

Remark 3.1.12. Quasi-isomorphism is not an equivalence relation because it does not satisfy the symmetric property. For example, consider the following quasi-isomorphism of chain complexes in the category of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

where $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the quotient map. There is only one morphism of chain complexes in the other direction, but it is not a quasi-isomorphism.

Lemma 3.1.13. Let $\mathcal{A}$ be an abelian category.

(1) If $f, g : A_\bullet \to B_\bullet$ are homotopic, then the induced maps $H_i(f)$ and $H_i(g)$ are equal.

(2) If $f : A_\bullet \to B_\bullet$ is a homotopy equivalence, then f is a quasi-isomorphism.

Proof.

(1) Let $h_i : A_i \to B_{i+1}$ be a collection of morphisms such that $f_i - g_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$. Then $(f_i - g_i)(\ker d_i) \subseteq \mathrm{im}(d_{i+1})$ because $h_{i-1}(d_i(\ker d_i)) = h_{i-1}(0) = 0$ and $d_{i+1}(h_i(\ker d_i)) \subseteq \mathrm{im}(d_{i+1})$. By definition of homology and the fact that the category $\mathsf{Ch}(\mathcal{A})$ is also abelian, $H_i(f - g) = H_i(f) - H_i(g) = 0$, which means $H_i(f) = H_i(g)$.

(2) Let $g : B_\bullet \to A_\bullet$ such that $g \circ f$ is homotopic to $\mathrm{id}_{A_\bullet}$ and $f \circ g$ is homotopic to $\mathrm{id}_{B_\bullet}$. Then $H_i(g) \circ H_i(f) = H_i(g \circ f) = H_i(\mathrm{id}_{A_\bullet}) = \mathrm{id}_{H_i(A_\bullet)}$, and $H_i(f) \circ H_i(g) = H_i(f \circ g) = H_i(\mathrm{id}_{B_\bullet}) = \mathrm{id}_{H_i(B_\bullet)}$ implies $H_i(f)$ is an isomorphism. So f is a quasi-isomorphism. $\square$

3.2 Differential and Naive Cotangent Complex

Definition 3.2.1. [Sta24, 00RN] Let $\varphi : R \to S$ be a ring homomorphism and M be an S-module. An R-**derivation** into M is an R-linear map $d : S \to M$ that satisfy $d(ab) = ad(b) + bd(a)$ (Leibniz rule).

The set of all R-derivations into M forms an S-module and is called the **module of derivation**, denoted by $\mathrm{Der}_R(S, M)$.

If $f : M \to N$ is an S-module homomorphism and $f : S \to M$ is an R-derivation into

M, then $f \circ d$ is an R-derivation into N. In this way, $\mathrm{Der}_R(S, -)$ is a functor from the category of S-modules to the category of S-modules.

Lemma 3.2.2. There exist S-module $\Omega_{S/R}$ with a S-module homomorphism $\mathrm{d} : S \to \Omega_{S/R}$ such that $\mathrm{Hom}(\Omega_{S/R}, M) \to \mathrm{Der}(S, M)$ defined by $\alpha \mapsto \alpha \circ \mathrm{d}$ gives an isomorphism of functors from $\mathrm{Hom}(\Omega_{S/R}, -)$ to $\mathrm{Der}(S, -)$.

$$\begin{tikzcd} S \arrow[r, "\mathrm{d}"] \arrow[dr, "\alpha\circ\mathrm{d}"'] & \Omega_{S/R} \arrow[d, "\alpha"] \\ & M \end{tikzcd}$$

Proof. Define the following map of free S-module

$$\left(\bigoplus_{(x,y)\in S^2} S[(a,b)]\right) \oplus \left(\bigoplus_{(f,g)\in S^2} S[(f,g)]\right) \oplus \left(\bigoplus_{(r\in R)} S[r]\right) \to \bigoplus_{a\in S} S[a]$$

defined by

$$\begin{aligned} [(a,b)] &\longmapsto [a+b]-[a]-[b] \\ [(f,g)] &\longmapsto [fg]-f[g]-g[f] \\ [r] &\longmapsto [\varphi(r)] \end{aligned}$$

Denote the cokernel of this map by $\Omega_{S/R}$. Then $\Omega_{S/R}$ satisfies the universal property claimed by construction. $\square$

Definition 3.2.3. The pair $(\Omega_{S/R}, \mathrm{d})$ is call the **module of differential** of S over R.

Lemma 3.2.4. [Sta24, 00RR] Suppose the following is a commutative diagram of rings

$$\begin{tikzcd} S \arrow[r, two heads, "\varphi"] & S' \\ R \arrow[u, "\alpha"] \arrow[r, "\psi"'] & R' \arrow[u, "\beta"'] \end{tikzcd}$$

where $\varphi : S \to S'$ is surjective with $\ker\varphi = I$. Then $\Omega_{S/R} \to \Omega_{S'/R'}$ is surjective with kernel generated as an S-modules by elements $\mathrm{d}a$, where $\varphi(a) \in \beta(R')$.

Lemma 3.2.5. [Sta24, 00RU] Suppose the following is a commutative diagram of rings

$$\begin{tikzcd} S \arrow[r, two heads, "\varphi"] & S' \\ R \arrow[u, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

where $\varphi : S \to S'$ is surjective with $\ker\varphi = I$. Then there is a canonical exact sequence of S'-modules

$$\begin{aligned} I/I^2 &\longrightarrow \Omega_{S/R} \otimes_S S' \longrightarrow \Omega_{S'/R} \longrightarrow 0 \\ f + I^2 &\longmapsto \mathrm{d}f \end{aligned}$$

Lemma 3.2.6. [Sta24, 02HP] Suppose the following is a commutative diagram of rings

$$\begin{tikzcd} S \arrow[r, two heads, "\varphi"] & S' \\ R \arrow[u, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

where $\varphi : S \to S'$ is surjective with $\ker\varphi = I$. Assume there exist R-algebra homomorphism $S' \to S$ which is a right inverse to φ. Then there is a canonical splitting short exact sequence of S'-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega_{S/R} \otimes_S S' & \longrightarrow & \Omega_{S'/R} & \longrightarrow & 0 \\ & & f + I^2 & \longmapsto & \mathrm{d}f & & & & \end{array}$$

Lemma 3.2.7. [Sta24, 02HQ] Let $R \to S$ be a ring homomorphism, $I \subseteq S$ be an ideal, and $n \in \mathbb{N}$. Let $S' = S/I^{n+1}$. Then the induced map $\Omega_{S/R} \to \Omega_{S'/R}$ induces an isomorphism

$$\Omega_{S/R} \otimes_S S/I^n \to \Omega_{S'/R} \otimes_{S'} S/I^n$$

Lemma 3.2.8. [Sta24, 00RX] If $S = R[x_i \mid i \in I]$, then $\Omega_{S/R}$ is a free S-modules with basis $\{\mathrm{d}x_i \mid i \in I\}$.

Definition 3.2.9. [Sta24, 07BN] Let $R \to S$ be a ring homomorphism. The **naive cotangent complex** $NL_{S/R}$ is the chain complex

$$NL_{S/R} = (I/I^2 \to \Omega_{R[S]/R} \otimes_{R[S]} S)$$

with I/I^2 placed in degree 1 and $\Omega_{R[S]/R} \otimes_{R[S]} S$ placed in degree 0.

Remark 3.2.10. There is an actual cotangent complex associated to a ring homomorphism. See Stacks Project 08PL.

Definition 3.2.11. Let $R \to S$ be a ring homomorphism. A **presentation** of S over R is a surjection $\alpha : P \to S$ of R-algebras where P is a polynomial algebra. For every presentation $\alpha : P \to S$ with $\ker\alpha = I$, we have a two term chain complex of S-modules

$$NL(\alpha) : I/I^2 \longrightarrow \Omega_{P/R} \otimes_P S$$

with I/I^2 placed in degree 1 and $\Omega_{P/R} \otimes_P S$ placed in degree 0. The complex $NL(\alpha)$ is called the **naive cotangent complex associated to the presentation** $\alpha : P \to S$.

Lemma 3.2.12. [Sta24, 00S1] Suppose the following is a commutative diagram of rings

$$\begin{tikzcd} S \arrow[r, "\varphi"] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd}$$

Let $\alpha : P \to S$ and $\alpha' : P' \to S'$ be presentations.

(1) There exists a morphism of presentation from α to α'.

(2) Any two morphisms of presentations induce homotopic morphism of complexes $NL(\alpha) \to NL(\alpha')$.

(3) The construction is compatible with compositions of morphisms of presentations.

(4) If $R \to R'$ and $S \to S'$ are isomorphisms, then for any morphism of presentation $\alpha \to \alpha'$, the induced map $NL(\alpha) \to NL(\alpha')$ is a homotopy equivalence and a quasi-isomorphism.

3.3 Smooth and Étale Maps

The goal of this section is to define smooth and étale morphism of schemes.

Definition 3.3.1. [Sta24, 00F3] Let $R \to S$ be a ring homomorphism

(1) $R \to S$ is of **finite type** if there exist $n \in \mathbb{N}$ and a surjection of R-algebras $R[x_1, \dots, x_n] \to S$.

(2) $R \to S$ is of **finite presentation** if there exist $n, m \in \mathbb{N}$ and polynomials $f_1, \dots, f_m \in R[x_1, \dots, x_n]$ and an isomorphism of R-algebras $R[x_1, \dots, x_n]/\langle f_1, \dots, f_m \rangle \simeq S$.

Definition 3.3.2. [Sta24, 00T2] A ring homomorphism $R \to S$ is **smooth** if it is of finite presentation and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module place in degree 0.

Remark 3.3.3. If $R \to S$ is a smooth ring homomorphism, then $\Omega_{S/R}$ is a finite projective S-module.

Lemma 3.3.4. [Sta24, 05GK] Let $R \to S$ be a ring homomorphism of finite presentation. Suppose $\alpha : P \to S$ is a presentation of S over R such that the naive cotangent complex $NL(\alpha)$ is quasi-isomorphic to a finite projective S-module placed in degree 0, then this holds for any presentation.

Definition 3.3.5. [Sta24, 00TI] A ring homomorphism $R \to S$ is **formally smooth** over R if for every commutative diagram

$$\begin{tikzcd} S \arrow[r] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

where A is a ring and $I \subseteq A$ is an ideal such that $I^2 = 0$, there exist a ring homomorphism $S \to A$ such that the diagram commute

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

Lemma 3.3.6. [Sta24, 031H] A composition of formally smooth ring homomorphisms is formally smooth.

Proof. Let $R \to S$ and $S \to T$ be formally smooth ring homomorphisms. Suppose given a commutative diagram

$$\begin{tikzcd} T \arrow[r] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

where A is a ring and $I \subseteq A$ is an ideal such that $I^2 = 0$, it suffices to show there exist $T \to A$ making the diagram commute. We have the commutative diagram on the left where $S \to A/I$ is the composition $S \to T \to A/I$

$$\begin{tikzcd} T \arrow[r] & A/I \\ S \arrow[u] \arrow[ur] & \\ R \arrow[u] \arrow[r] & A \arrow[uu] \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r] & A/I \\ S \arrow[u] \arrow[ur] \arrow[dr, dashed] & \\ R \arrow[u] \arrow[r] & A \arrow[uu] \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r] \arrow[ddr, dashed] & A/I \\ S \arrow[u] \arrow[ur] \arrow[dr, dashed] & \\ R \arrow[u] \arrow[r] & A \arrow[uu] \end{tikzcd}$$

Since $R \to S$ is formally smooth, there exist ring homomorphism $S \to A$ giving the middle commutative diagram. $S \to T$ is formally smooth implies there exist ring homomorphism $T \to A$ giving the right commutative diagram. Then $T \to A$ lifts $T \to A/I$ which shows $R \to S \to T$ is formally smooth. □

Lemma 3.3.7. [Sta24, 00TK] A polynomial ring over R is formally smooth over R.

Proof. Let $P = R[X_i \mid i \in I]$ be a polynomial ring over R. Suppose given a commutative diagram

$$\begin{tikzcd} P \arrow[r] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

where $R \to P$ is the inclusion map, A is a ring, and $I \subseteq A$ is an ideal such that $I^2 = 0$. $P \to A/I$ can be lifted to a ring homomorphism $P \to A$ by mapping generators to a lift in A and using universal property of polynomial ring. So $R \to P$ is formally smooth. □

Lemma 3.3.8. [Sta24, 00TL] Let $R \to S$ be a ring homomorphism. Let $P \to S$ be a presentation of S over R. Denote $J \subseteq P$ the kernel. Then $R \to S$ is formally smooth if and only if there exists an R-algebra homomorphism $\sigma : S \to P/J^2$ which is a right inverse to the surjection $P/J^2 \to S$.

Remark 3.3.9. The proof of lemma 3.3.8 only make use of the fact that P is formally smooth over R, so the statement can be generalized. But we will only use it when P is a polynomial ring with coefficients in R.

Lemma 3.3.10. [Sta24, 031I] Let $R \to S$ be a ring homomorphism. Let $P \to S$ be a presentation of S over R. Denote $J \subseteq P$ the kernel. Then $R \to S$ is formally smooth if and only if the following is a splitting short exact sequence

$$0 \longrightarrow J/J^2 \longrightarrow \Omega_{P/R} \otimes_P S \longrightarrow \Omega_{S/R} \longrightarrow 0$$

Proposition 3.3.11. [Sta24, 031J] Let $R \to S$ be a ring homomorphism. The following are equivalent

(1) S is formally smooth over R,

(2) there exist presentation $P \to S$ of S over R with kernel $J \subseteq P$ such that there exists a section $S \to P/J^2$.

(3) for every presentation $P \to S$ of S over R with kernel $J \subseteq P$, there exist a section $S \to P/J^2$.

(4) there exist presentation $P \to S$ of S over R with kernel $J \subseteq P$ such that the following is a splitting short exact sequence

$$0 \longrightarrow J/J^2 \longrightarrow \Omega_{P/R} \otimes_P S \longrightarrow \Omega_{S/R} \longrightarrow 0$$

(5) for every presentation $P \to S$ of S over R with kernel $J \subseteq P$, the following is a splitting short exact sequence

$$0 \longrightarrow J/J^2 \longrightarrow \Omega_{P/R} \otimes_P S \longrightarrow \Omega_{S/R} \longrightarrow 0$$

(6) the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a projective S-module placed in degree 0.

Proposition 3.3.12. [Sta24, 00TN] Let $R \to S$ be a ring homomorphism. The following are equivalent

(1) $R \to S$ is of finite presentation and formally smooth,

(2) $R \to S$ is smooth.

Proof.

$(1) \implies (2)$ Suppose $R \to S$ is formally smooth, by proposition 3.3.11, then the naive cotangent complex is quasi-isomorphic to a projective S-module placed in degree 0. Since $R \to S$ is also of finite presentation, it follows that $R \to S$ is smooth by definition.

$(2) \implies (1)$ Suppose $R \to S$ is smooth, then the naive cotangent complex is quasi-isomorphic to a projective S-module placed in degree 0. by proposition 3.3.11, $R \to S$ is formally smooth. $R \to S$ is of finite presentation because it is smooth. □

Definition 3.3.13. [Sta24, 00U1] A ring homomorphism $R \to S$ is **étale** if it is of finite presentation and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to zero. Given a prime ideal $\mathfrak{q} \in \operatorname{Spec} S$, $R \to S$ is **étale at $\mathfrak{q}$** if there exists $g \in S \setminus \mathfrak{q}$ such that $R \to S_g$ is étale.

Definition 3.3.14. [Sta24, 01V5] Let $f : X \to S$ be a morphism of schemes.

(1) f is **smooth at $x \in X$** if there exist an affine open neighborhood $\operatorname{Spec} A = U \subseteq X$ of x and affine open $\operatorname{Spec} R = V \subseteq S$ with $f(U) \subseteq V$ such that the induced ring homomorphism $R \to A$ is smooth.

(2) f is **smooth** if it is smooth at every point of X.

Definition 3.3.15. [Sta24, 02GI] Let $f : X \to S$ be a morphism of schemes.

(1) f is **étale at $x \in X$** if there exist an affine open neighborhood $\operatorname{Spec} A = U \subseteq X$ of x and affine open $\operatorname{Spec} R = V \subseteq S$ with $f(U) \subseteq V$ such that the induced ring homomorphism $R \to A$ is étale.

(2) f is **étale** if it is étale at every point of X.

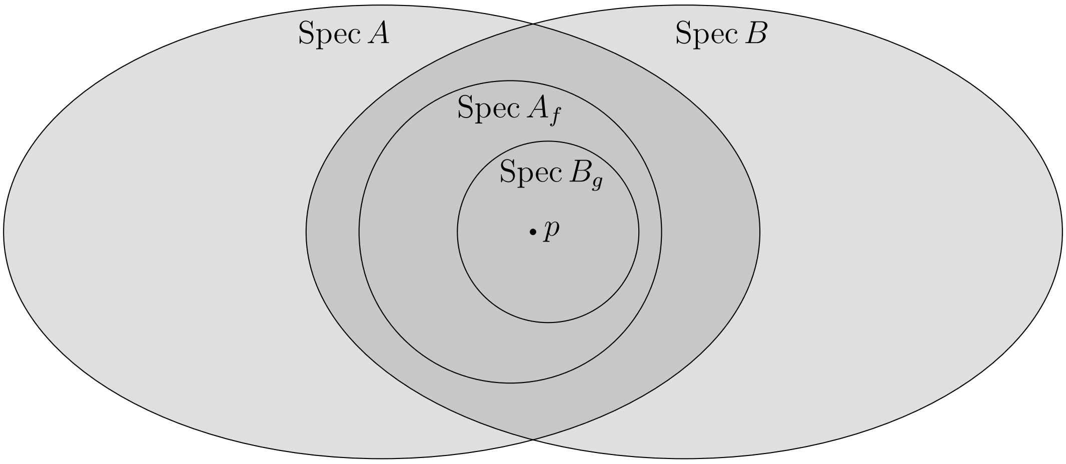
Lemma 3.3.16. Suppose $\operatorname{Spec} A$ and $\operatorname{Spec} B$ are affine open subschemes of a scheme X. Then $\operatorname{Spec} A \cap \operatorname{Spec} B$ is the union of open sets that are simultaneously basic open set of $\operatorname{Spec} A$ and $\operatorname{Spec} B$.

Proof. It suffices to show that for each point in $\operatorname{Spec} A \cap \operatorname{Spec} B$, there is an open set containing that point and is simultaneously basic open set of $\operatorname{Spec} A$ and $\operatorname{Spec} B$. Let $p \in \operatorname{Spec} A \cap \operatorname{Spec} B$. Since basic open sets $\{U_f : f \in A\}$ form a basis of $\operatorname{Spec} A$, and $\operatorname{Spec} A \cap \operatorname{Spec} B$ is an open subset of $\operatorname{Spec} A$, it follows that $\exists f \in A$ such that $U_f \subseteq \operatorname{Spec} A \cap \operatorname{Spec} B$. Moreover, $U_f = \operatorname{Spec} A_f$ (as schemes, where U_f have the open subscheme structure) because

$$\mathscr{O}_X(U_f) = \mathscr{O}_{\operatorname{Spec} A}(U_f) = A_f$$

Since basic open sets $\{U_g : g \in B\}$ form a basis of $\operatorname{Spec} B$, and $U_f = \operatorname{Spec} A_f$ is an open subset of $\operatorname{Spec} B$, it follows that $\exists g \in B$ such that $U_g \subseteq U_f = \operatorname{Spec} A_f$. Moreover, $U_g = \operatorname{Spec} B_g$ (as schemes, where U_g have the open subscheme structure) because

$$\mathscr{O}_X(U_g) = \mathscr{O}_{\operatorname{Spec} B}(U_g) = B_g$$



Consider the following restriction map

$$\begin{tikzcd}
\mathscr{O}_X(\operatorname{Spec} B) \arrow[rr] \arrow[d, equal] & & \mathscr{O}_X(\operatorname{Spec} A_f) \arrow[d, equal] \\
\mathscr{O}_{\operatorname{Spec} B}(\operatorname{Spec} B) \arrow[d, equal] & & \mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A_f) \arrow[d, equal] \\
B & & A_f \\
g \arrow[rr, mapsto] & & g'
\end{tikzcd}$$

and define $g' \in A_f$ to be the image of $g \in B$. Then

$$\begin{aligned}
\operatorname{Spec} B_g &= \{\mathfrak{p} \in \operatorname{Spec} B \mid g \notin \mathfrak{p}\} \\
&= \{\mathfrak{p} \in \operatorname{Spec} B \mid g(\mathfrak{p}) \notin \mathfrak{p}B_{\mathfrak{p}}\} \\
&= \{\mathfrak{q} \in \operatorname{Spec} A_f \mid g'(\mathfrak{q}) \notin \mathfrak{q}(A_f)_{\mathfrak{q}}\} \\
&= \{\mathfrak{q} \in \operatorname{Spec} A_f \mid g' \notin \mathfrak{q}\} = \operatorname{Spec}(A_f)_{g'}
\end{aligned}$$

If $g' = g''/f^n \in A_f$ with $g'' \in A$, then $\operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fg''}$. So $\operatorname{Spec} B_g = \operatorname{Spec} A_{fg''}$ is simultaneously basic open set of $\operatorname{Spec} A$ and $\operatorname{Spec} B$. $\square$

Lemma 3.3.17. (Affine Communication Lemma) Let $\mathcal{P}$ be some property enjoyed by some affine open subsets of a scheme X, such that for any affine open subset $\operatorname{Spec} A \subseteq X$

(i) if $\operatorname{Spec} A \subseteq X$ has property $\mathcal{P}$, then for any $f \in A$, $\operatorname{Spec} A_f \subseteq X$ does too

(ii) if $\langle f_1, \ldots, f_n \rangle = A$ and $\operatorname{Spec} A_{f_i} \subseteq X$ has $\mathcal{P}$ for all $i \in \{1, \ldots, n\}$, then so does $\operatorname{Spec} A \subseteq X$.

Suppose X has an open cover $\{\operatorname{Spec} A_i\}_{i \in I}$ each having $\mathcal{P}$. Then every affine open subset of X have $\mathcal{P}$.

Proof. Let $\operatorname{Spec} B \subseteq X$ be an affine open subset, it suffices to show B have $\mathcal{P}$. Then $\{\operatorname{Spec} A_i \cap \operatorname{Spec} B\}_{i \in I}$ covers $\operatorname{Spec} B$. For each $i \in I$, $\operatorname{Spec} A_i \cap \operatorname{Spec} B$ is a union of open sets that are simultaneously basic open set of $\operatorname{Spec} A_i$ and $\operatorname{Spec} B$ (lemma 3.3.16). Say

$$\operatorname{Spec} A_i \cap \operatorname{Spec} B = \bigcup_{j \in J_i} \operatorname{Spec}(A_i)_{f_j}$$

Then $\operatorname{Spec} B = \bigcup_{i \in I} \bigcup_{j \in J} \operatorname{Spec}(A_i)_{f_j}$. Since $\operatorname{Spec} B$ is quasi-compact, $\operatorname{Spec} B$ is a finite union of $\operatorname{Spec}(A_i)_{f_j}$. Say

$$\operatorname{Spec} B = \bigcup_{k=1}^{n} \operatorname{Spec}(A_{i_k})_{f_{j_k}}$$

By lemma 3.3.16, $\forall k \in \{1, \ldots, n\}$, $\operatorname{Spec}(A_{i_k})_{f_{j_k}} = \operatorname{Spec} B_{g_k}$ for some $g_k \in B$. Then

$$\operatorname{Spec} B = \bigcup_{k=1}^{n} \operatorname{Spec} B_{g_k}$$

Then $\langle g_1, \ldots, g_k \rangle = B$ because if not, then $\langle b_1, \ldots, b_k \rangle$ is contained in some maximal ideal that contains $g_1, \ldots, g_k$, contradicting the fact that $\operatorname{Spec} B$ is the union of $\operatorname{Spec} B_{g_k}$.

- By assumption (i), $\forall k \in \{1, \ldots, n\}$, $\operatorname{Spec} A_{i_k}$ have P implies $\operatorname{Spec}(A_{i_k})_{f_{j_k}} = \operatorname{Spec} B_{g_k}$ have $\mathcal{P}$.
- By assumption (ii), $\operatorname{Spec}(A_{i_k})_{f_{j_k}} = \operatorname{Spec} B_{g_k}$ have P for all $k \in \{1, \ldots, n\}$ implies $\operatorname{Spec} B$ have $\mathcal{P}$.

□

Lemma 3.3.18. [Sta24, 01V6] Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) The morphism f is smooth.

(2) For every affine opens $U \subseteq X$, $V \subseteq S$ with $f(U) \subseteq V$, the ring map $\mathscr{O}_S(V) \to \mathscr{O}_X(U)$ is smooth.

(3) There exist open covering $S = \bigcup_{i \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $U_i \to V_j$ is smooth for all $j \in J$ and $i \in I_j$.

(4) There exist an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathscr{O}_S(V_j) \to \mathscr{O}_X(U_i)$ is smooth for all $j \in J$ and $i \in I_j$.

Proof. It is clear that (2) $\implies$ (4) $\implies$ (3) $\implies$ (1). So it suffices to show (1) $\implies$ (2). By Affine Communication Lemma (lemma 3.3.17), it suffices to show

(i) If $B \to A$ is a smooth ring homomorphism, then for all $f \in A$, $B \to A_f$ is a smooth ring homomorphism.

(ii) If $B \to A$ is a ring homomorphism such that $f_1, \ldots, f_n \in A$, $B \to A_{f_i}$ are smooth ring homomorphisms, and $\langle f_1, \ldots, f_n \rangle = A$, then $B \to A$ is a smooth ring homomorphism.

Both of these statements are algebra results. □

Lemma 3.3.19. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) The morphism f is étale.

(2) For every affine opens $U \subseteq X$, $V \subseteq S$ with $f(U) \subseteq V$, the ring map $\mathscr{O}_S(V) \to \mathscr{O}_X(U)$ is étale.

(3) There exist open covering $S = \bigcup_{i \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $U_i \to V_j$ is étale for all $j \in J$ and $i \in I_j$.

(4) There exist an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathscr{O}_S(V_j) \to \mathscr{O}_X(U_i)$ is étale for all $j \in J$ and $i \in I_j$.

Proof. It is clear that (2) $\implies$ (4) $\implies$ (3) $\implies$ (1). So it suffices to show (1) $\implies$ (2). By lemma 3.3.17, it suffices to show

(i) If $B \to A$ is a étale ring homomorphism, then for all $f \in A$, $B \to A_f$ is a étale ring homomorphism.

(ii) If $B \to A$ is a ring homomorphism such that $f_1, \ldots, f_n \in A$, $B \to A_{f_i}$ are étale ring homomorphisms, and $\langle f_1, \ldots, f_n \rangle = A$, then $B \to A$ is a étale ring homomorphism.

Both of these statements are algebra results. □

4 Algebraic Space and Algebraic Stack

4.1 Yoneda Lemma and Representability

The goal of this section is to understand what it means for a natural transformation to be representable.

Definition 4.1.1. [Sta24, 001O] Let $\mathcal{C}$ be a category. For any $U \in \mathrm{Ob}(\mathcal{C})$

$$\begin{aligned} h_U : \mathcal{C} &\longrightarrow \mathsf{Sets} \\ V &\longmapsto \mathrm{Mor}_{\mathcal{C}}(V, U) \\ (f : W \longrightarrow V) &\longmapsto \left(\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(V, U) &\longrightarrow \mathrm{Mor}_{\mathcal{C}}(W, U) \\ g &\longmapsto g \circ f \end{aligned} \right) \end{aligned}$$

defines a contravariant functor. It is called the **representable presheaf** (definition 2.2.2) associated to U. This functor is denoted by h_U.

Definition 4.1.2. [Sta24, 001Q] Let $\mathcal{C}$ be a category. A contravariant functor $F : \mathcal{C} \to \mathsf{Sets}$ is **representable** if it is isomorphic to h_U for some $U \in \mathrm{Ob}(\mathcal{C})$.

Lemma 4.1.3. [Sta24, 001P] (Yoneda lemma) Let $\mathcal{C}$ be a category and F be contravariant functor from $\mathcal{C}$ to Sets. Then for every $U \in \mathrm{Ob}(\mathcal{C})$, there is a natural bijection between the natural transformation $\eta : h_U \to F$ and the set $F(U)$.

Proof. Denote the set of natural transformation $h_U \to F$ by $\mathrm{Nat}(h_U, F)$.

Bijection. Define $\Phi : \mathrm{Nat}(h_U, F) \to F(U)$ by $\Phi(\eta) = \eta_U(\mathrm{id}_U)$ for $\eta \in \mathrm{Nat}(h_U, F)$. Define $\Psi : F(U) \to \mathrm{Nat}(h_U, F)$ by $\Psi(x)_V(f) = F(f)(x)$ ($f \in h_U(V) = \mathrm{Mor}_{\mathcal{C}}(V, U)$) where $x \in F(U)$ To check Ψ is well-defined, it suffices to show $\Psi(x)$ is a natural transformation. Let $V, W \in \mathrm{Ob}(\mathcal{C})$ and $f : V \to W$ a morphism. Then the following diagram commute

$$\begin{tikzcd} h_U(V) \arrow[r, "\Psi(x)_V"] \arrow[d, "- \circ f"'] & F(V) \arrow[d, "F(f)"] \\ h_U(W) \arrow[r, "\Psi(x)_W"'] & F(W) \end{tikzcd}$$

because given $g \in h_V(U) = \mathrm{Mor}(V, U)$

$$F(f)(\Psi(x)_V(g)) = F(f)(F(g)(x)) = (F(f) \circ F(g))(x) = F(g \circ f)(x) = \Psi(x)_W(g \circ f)$$

Φ and Ψ are inverses of each other because

$$\Phi(\Psi(x)) = \Psi(x)_U(\mathrm{id}_U) = F(\mathrm{id}_U)(x) = \mathrm{id}_{F(U)}(x) = x$$
$$\Psi(\Phi(\eta))_V(f) = \Psi(\eta_U(\mathrm{id}_U))_V(f) = F(f)(\eta_U(\mathrm{id}_U)) = \eta_V(\mathrm{id}_U \circ f) = \eta_V(f)$$

Naturality. Let $f : V \to U$, it suffices to show the following diagram commute

$$\begin{tikzcd}
\mathrm{Nat}(h_U, F) \arrow[r, "\Phi^U"] \arrow[d, "-\circ\Psi(f)"'] & F(U) \arrow[d, "F(f)"] \\
\mathrm{Nat}(h_V, F) \arrow[r, "\Phi^V"'] & F(V)
\end{tikzcd}$$

where $\Psi : h_U(V) \to \mathrm{Nat}(h_V, h_V)$ is defined by $\Psi(f) = - \circ f$. This is the case because for $\eta \in \mathrm{Nat}(h_U, F)$

$$F(f)\Phi^U(\eta) = F(f)\eta_U(\mathrm{id}_U) = \eta_V(f \circ \mathrm{id}_U) = \eta_V(f)$$
$$\Phi^V(\eta \circ \Psi(f)) = (\eta \circ \Psi(f))_V(\mathrm{id}_V) = \eta_V\Psi(f)_V(\mathrm{id}_V) = \eta_V(\mathrm{id}_V \circ f) = \eta_V(f)$$

where the second equality hold because the diagram in the construction of bijection in this proof is commutative. Therefore, the bijection between $\mathrm{Nat}(h_U, F)$ and $F(U)$ is natural. □

Remark 4.1.4. Let $\mathcal{C}$ be a category and $U, V \in \mathrm{Ob}(\mathcal{C})$. By Yoneda lemma (lemma 4.1.3) applied to the functor h_V, there exist a natural bijection between the natural transformations $h_U \to h_V$ and $h_U(V) = \mathrm{Mor}_{\mathcal{C}}(V, U)$. This implies an object determines and is determined by its representable functor.

Definition 4.1.5. In the category of schemes, a representable functor $h_X : \mathsf{Sch} \to \mathsf{Set}$ is called the **functor of points** of the scheme X. For a scheme Y, $h_X(Y) = \mathrm{Mor}_{\mathsf{Sch}}(Y, X)$ is called the **Y-points of X**, and a morphism of scheme $Y \to X$ is called a Y-point of X. This terminology generalizes the terminology from ring theory which we explain in the remark below.

Remark 4.1.6. Let $f \in \mathbb{Z}[x_1, \dots, x_n]$ and R be a ring. The R-points of f is defined to be $\{(a_1, \dots, a_n) \in R^n \mid f(a_1, \dots, a_n) = 0\}$ $R \mapsto \{(a_1, \dots, a_n) \in R^n \mid f(a_1, \dots, a_n) = 0\}$ defines a covariant functor from the category of rings to the category of sets. Then

$$\begin{aligned}
\{(a_1, \dots, a_n) \in R^n \mid f(a_1, \dots, a_n) = 0\} &= \mathrm{Mor}_{\mathsf{Rings}}\left(\frac{\mathbb{Z}[x_1, \dots, x_n]}{\langle f \rangle}, R\right) \\
&= \mathrm{Mor}_{\mathsf{Sch}}\left(\mathrm{Spec}\, R, \mathrm{Spec}\, \frac{\mathbb{Z}[x_1, \dots, x_n]}{\langle f \rangle}\right)
\end{aligned}$$

This set of morphism is called the R-points of f, so when $\mathrm{Spec}\, R$ is replaced with Y and $\mathrm{Spec}(\mathbb{Z}[x_1, \dots, x_n]/\langle f \rangle)$ is replaced with X, we call $\mathrm{Mor}_{\mathsf{Sch}}(Y, X)$ the Y-points of X.

Lemma 4.1.7. [Sta24, 0022] Let $\mathcal{C}$ be a category, F, G, H be contravariant functor from $\mathcal{C}$ to Sets, and $a : F \to G$, $b : H \to G$ be natural transformations. Then $F \times_{a,G,b} H$ defined as follows

$$(F \times_{a,G,b} H)(U) = F(U) \times_{a_U, G(U), b_U} H(U)$$

for any $U \in \mathrm{Ob}(\mathcal{C})$ is also a contravariant functor. $F \times_{a,G,b} H$ also defines the fiber product of F, H over G in the category $\mathsf{PSh}(\mathcal{C})$.

Proof.

$F \times_G H$ is a contravariant functor. Let $U, V \in \mathrm{Ob}(\mathcal{C})$ and $\varphi \in \mathrm{Mor}_{\mathcal{C}}(U, V)$. By universal property of fiber products, there exist an induced map $(F \times_G H)(V) \to (F \times_G H)(U)$.

$$\begin{tikzcd}
(F \times_G H)(V) \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & H(V) \arrow[dr, "H(\varphi)"] \arrow[dd, "b_V" near start] & \\
& (F \times_G H)(U) \arrow[rr, crossing over] & & H(U) \arrow[dd, "b_U"] \\
F(V) \arrow[rr, "a_V" near start] \arrow[dr, "F(\varphi)"'] & & G(V) \arrow[dr, "G(\varphi)"] & \\
& F(U) \arrow[rr, "a_U"'] \arrow[from=uu, crossing over] & & G(U)
\end{tikzcd}$$

$F \times_G H$ respects compositions and identity by the uniqueness of the induced map from the universal property of fiber product.

$F \times_G H$ satisfies the universal property of fiber product $F \times_G H$ comes with maps $p_F : F \times_G H \to F$ and $p_H : F \times_G H \to H$ defined by

Suppose $K : \mathcal{C} \to \mathsf{Sets}$ be a contravariant functor with natural transformation $c : K \to F$, $d : K \to H$ $\square$

Definition 4.1.8. [Sta24, 0023] Let $\mathcal{C}$ be a category and F, G be contravariant functors from $\mathcal{C}$ to Sets. A natural transformation $a : F \to G$ is **representable** if for every $U \in \mathrm{Ob}(\mathcal{C})$ and every $\xi \in G(U)$, the functor $F \times_G h_U$ is representable

$$\begin{tikzcd}
& F \times_G h_U \arrow[dl] \arrow[dr] & \\
F \arrow[dr, "a"'] & & h_U \arrow[dl, "\xi"] \\
& G &
\end{tikzcd}$$

using the natural bijection in Yoneda lemma (lemma 4.1.3), $\xi \in G(U)$ correspond to a natural transformation $h_U \to G$ which we will also denote by ξ.

Lemma 4.1.9. [Sta24, 0024] Let $\mathcal{C}$ be a category and F be a contravariant functor from $\mathcal{C}$ to Sets. Assume $\mathcal{C}$ has products of pairs of objects and fiber products. Then the following are equivalent

(1) The diagonal $\Delta : F \to F \times F$ is representable.

(2) For every $U \in \mathrm{Ob}(\mathcal{C})$, and any $\xi \in F(U)$, the map $\xi : h_U \to F$ is representable.

(3) For every $U, V \in \mathrm{Ob}(\mathcal{C})$, and any $\xi \in F(U), \xi' \in F(V)$, the fiber product $h_U \times_F h_V$ is representable.

Proof.

(2) $\iff$ (3) Unwinding the definition for a natural transformation to be representable in (2) (definition 4.1.8) shows it is precisely (3).

(1) $\implies$ (3) Suppose $\Delta : F \to F \times F$ is representable. Let $U, V \in \mathrm{Ob}(\mathcal{C})$ and $\xi \in F(U), \xi' \in F(V)$. It suffices to show the functor $h_U \times_F h_V$ is representable.

By the assumption of existence of product, $U \times V \in \mathrm{Ob}(\mathcal{C})$. By lemma 4.1.7, $h_U \times h_V = h_{U\times V}$ which is representable. Let $\xi \times \xi' : h_{U\times V} \to F \times F$ be the morphism induced by the universal property of product of $F \times F$

$$\begin{tikzcd}
 & h_U \times h_V \arrow[dl] \arrow[dr] \arrow[d, equal] & \\
h_U \arrow[dd, "\xi"'] & h_{U\times V} \arrow[d, dashed, "\xi\times\xi'"] & h_V \arrow[dd, "\xi'"] \\
 & F \times F \arrow[dl] \arrow[dr] & \\
F & & F
\end{tikzcd}$$

By lemma 4.1.3, suppose $W \in \mathrm{Ob}(\mathcal{C})$ and $(\varphi, \varphi') \in h_U(W) \times h_V(W) = h_{U\times V}$, then

$$(\xi \times \xi')(\varphi, \varphi') = (\xi(\varphi), \xi'(\varphi')) = (F(\varphi)(\xi), F(\varphi')(\xi'))$$

Let $W \in \mathrm{Ob}(\mathcal{C})$, then

$$\begin{aligned}
(F \underset{F\times F}{\times} h_{U\times V})(W) &= F(W) \underset{(F\times F)(W)}{\times} h_{U\times V}(W) \\
&= \left\{ \begin{matrix} (\varphi, \varphi') \in h_{U\times V}(W) \\ \theta \in F(W) \end{matrix} \;\middle|\; \Delta(\theta) = (\xi \times \xi')(\varphi, \varphi') \right\} \\
&= \left\{ \begin{matrix} \varphi \in h_U(W) \\ \varphi' \in h_V(W) \\ \theta \in F(W) \end{matrix} \;\middle|\; (\theta, \theta) = (F(\varphi)(\xi), F(\varphi')(\xi')) \right\} \\
&= \left\{ \begin{matrix} \varphi \in h_U(W) \\ \varphi' \in h_V(W) \end{matrix} \;\middle|\; F(\varphi)(\xi) = F(\varphi')(\xi') \right\} \\
&= h_U(W) \underset{F(W)}{\times} h_V(W) = (h_U \times_F h_V)(W)
\end{aligned}$$

By definition of $\Delta : F \to F \times F$ is representable, $F \times_{F\times F} h_{U\times V}$ is a representable functor. Therefore, $h_U \times_F h_V = F \times_{F\times F} h_{U\times V}$ is a representable functor.

(3) $\implies$ (1) Suppose for every $U, V \in \mathrm{Ob}(\mathcal{C})$, and any $\xi \in F(U)$, $\xi' \in F(V)$, the fiber product $h_U \times_F h_V$ is representable. By definition 4.1.8, it suffices to show for every $U \in \mathrm{Ob}(\mathcal{C})$ and every $(\xi, \xi') \in (F \times F)(U) = F(U) \times F(U)$, the functor $F \times_{F\times F} h_U$ is

representable.

$$\begin{aligned}(F \underset{F\times F}{\times} h_U)(W) &= F(W) \underset{F(U)\times F(U)}{\times} h_U(W) \\ &= \left\{ \begin{array}{c} \varphi \in h_U(W) \\ \theta \in F(W) \end{array} \middle| \Delta(\theta) = (\xi, \xi')(\varphi) \right\} \\ &= \left\{ \begin{array}{c} \varphi \in h_U(W) \\ \theta \in F(W) \end{array} \middle| (\theta, \theta) = ((F(\varphi)(\xi)), (F(\varphi)(\xi'))) \right\} \\ &= h_U(W) \times_{F(W)} h_U(W) = (h_U \times_F h_U)(W)\end{aligned}$$

where the functor $h_U \times_F h_U$ comes from $\xi \in F(U)$ and $\xi' \in F(U)$ By assumption, $h_U \times_F h_U$ is representable which implies $F \times_{F\times F} h_U = h_U \times_F h_U$ is representable. By definition 4.1.8, $\Delta : F \to F \times F$ is representable. □

Remark 4.1.10. lemma 4.1.9 also follows from the diagonal base change diagram (One reference is [Vak24, Exercise 1.3.S.]) which states in any category, assuming relevant fiber product exist, the following is a fiber product diagram

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times X_2 \arrow[d] \\ Y \arrow[r, "\Delta"'] & Y \times Y \end{tikzcd}$$

or more generally, the following is a fiber product diagram

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r, "\Delta"'] & Y \times_Z Y \end{tikzcd}$$

The proof is a rather lengthy diagram chase, and we do not need the result here.

4.2 fppf Topology

Definition 4.2.1. Let $f : X \to S$ be a morphism of schemes.

(1) $f : X \to S$ is **quasi-compact** if the underlying map of topological spaces is quasi-compact, i.e., if $V \subseteq S$ is quasi-compact, then $f^{-1}(V)$ is quasi-compact.

(2) $f : X \to S$ is **quasi-separated** if the diagonal morphism $\Delta_{X/S} : X \to X \times_S X$ is quasi-compact.

(3) f is **flat at a point** $x \in X$ if the local ring $\mathscr{O}_{X,x}$ is flat over the local ring $\mathscr{O}_{S,f(x)}$, i.e. $\mathscr{O}_{X,x}$ is a flat $\mathscr{O}_{S,f(x)}$-module.

(4) f is **flat** if f is flat at every point of X.

(5) f is of **finite presentation at** $x \in X$ if there exist an affine open neighborhood $\operatorname{Spec} A = U \subseteq X$ of x and affine open $\operatorname{Spec} R = V \subseteq S$ with $f(U) \subseteq V$ such that

the induced ring homomorphism $R \to A$ is of finite presentation (definition 3.3.1).

(6) f is **locally of finite presentation** if it is of finite presentation at every point of X.

(7) f is **of finite presentation** if it is locally of finite presentation, quasi-compact and quasi-separated.

Definition 4.2.2. [Sta24, 021M] Let T be a scheme. An **fppf covering** of T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is flat, locally of finite presentation and such that $T = \bigcup f_i(T_i)$.

Remark 4.2.3. The letters fppf stand for "fidèlement plat de présentation finie".

Lemma 4.2.4. [Sta24, 021O] Let T be a scheme.

(1) If $T' \to T$ is an isomorphism, then $\{T' \to T\}$ is an fppf covering of T.

(2) If $\{T_i \to T\}_{i \in I}$ is an fppf covering and for each i, we have an fppf covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fppf covering.

(3) If $\{T_i \to T\}_{i \in I}$ is an fppf covering and $T' \to T$ is a morphism of schemes, then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fppf covering.

Definition 4.2.5. [Sta24, 021R] A **big fppf site** is any site Sch_{fppf} constructed as follows:

(1) Choose any set of schemes S_0, and any set of fppf coverings Cov_0 among theses schemes.

(2) As underlying category, take any category Sch_α starting with the set S_0.

(3) Choose any set of coverings starting with the category Sch_α and the class of fppf coverings, and the set Cov_0.

Remark 4.2.6. The idea behind Sch_α is that it is a category where the collection of objects is a set, rather than a proper class, of schemes, and it is closed under a list of natural operations. Then $\mathrm{Sch}_{\mathrm{fppf}}$ is giving a site structure to the category Sch_α.

Definition 4.2.7. [Sta24, 021S] Let S be a scheme. Let $\mathrm{Sch}_{\mathrm{fppf}}$ be a big fppf site containing S. The **big fppf site of S**, denoted by $(\mathrm{Sch}/S)_{\mathrm{fppf}}$, is the site Sch_{fppf}/S (item (3)).

4.3 Algebraic Space

The goal of this section is to define an algebraic space over a base scheme.

Definition 4.3.1. Let S be a scheme contained in $\mathrm{Sch}_{\mathrm{fppf}}$. Let $F, G : (\mathrm{Sch}/S)_{\mathrm{fppf}}^{\mathrm{opp}} \to \mathsf{Sets}$ be presheaves, and $a : F \to G$ be a representable transformation of functors (definition 4.1.8). By definition of representable transformation, for every $U \in \mathrm{Ob}((\mathrm{Sch}/S)_{\mathrm{fppf}})$

and any $\xi \in G(U)$, the fiber product $h_U \times_{\xi,G,a} F$ is representable. Choose a representing object V_ξ and an isomorphism $h_{V_\xi} \to h_U \times_G F$. By Yoneda lemma (lemma 4.1.3), the projection $h_{V_\xi} \to h_U \times_G F \to h_U$ comes from a unique morphism of schemes $a_\xi : V_\xi \to U$.

$$\begin{tikzcd}
V_\xi \arrow[r, squiggly] \arrow[d, "a_\xi"'] & h_{V_\xi} \arrow[r] \arrow[d] & F \arrow[d, "a"] \\
U \arrow[r, squiggly] & h_U \arrow[r, "\xi"'] & G
\end{tikzcd}$$

Let $\mathcal{P}$ be a property of schemes which

(1) is preserved under any base change, and

(2) is fppf local on the base.

In this case, we say that a has property $\mathcal{P}$ if for every $U \in \mathrm{Ob}(\mathrm{Sch}/S)_{\mathrm{fppf}}$ and any $\xi \in G(U)$, the resulting morphism of schemes $V_\xi \to U$ has property $\mathcal{P}$.

Definition 4.3.2. [Sta24, 025Y] Let S be a scheme contained in $\mathrm{Sch}_{\mathrm{fppf}}$. An **algebraic space** over S is a presheaf

$$F : (\mathrm{Sch}/S)^{\mathrm{opp}}_{\mathrm{fppf}} \to \mathsf{Sets}$$

with the following properties

(1) The presheaf F is a sheaf (definition 2.2.3).

(2) The diagonal morphism $F \to F \times F$ is representable (definition 4.1.8).

(3) There exists a scheme $U \in \mathrm{Ob}((\mathrm{Sch}/S)_{\mathrm{fppf}})$ and a map $h_U \to F$ which is surjective and étale. (definition 4.3.1)

Lemma 4.3.3. [Sta24, 025Z] A scheme is an algebraic space. More precisely, given a scheme $T \in \mathrm{Ob}((\mathrm{Sch}/S)_{\mathrm{fppf}})$, the representable presheaf h_T is an algebraic space.

Proof. We check h_T satisfies the three conditions in definition 4.3.2. In the site $\mathrm{Sch}_{\mathrm{fppf}}$, all representable presheaves are sheaves. The diagonal morphism $h_T \to h_T \times h_T = h_{T\times_S T}$ is representable because the fiber product $T \times_S T$ exist in $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. The identity map $h_T \to h_T$ is surjective and étale. □

4.4 Algebraic Stack

The goal of this section is to define an algebraic stack over a base scheme.

Definition 4.4.1. [Sta24, 02ZQ] Let S be a scheme contained in $\mathrm{Sch}_{\mathrm{fppf}}$. Let $p : \mathcal{X} \to (\mathrm{Sch}/S)_{\mathrm{fppf}}$ be a category fibered in groupoids (definition 2.3.2) over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. $\mathcal{X}$ is **representable by a scheme** if there exist a scheme $U \in \mathrm{Ob}((\mathrm{Sch}/S)_{\mathrm{fppf}})$ and an equivalence

$$j : \mathcal{X} \longrightarrow (\mathrm{Sch}/U)_{\mathrm{fppf}}$$

of categories over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.

Definition 4.4.2. [Sta24, 04SV] Let S be a scheme contained in $\mathrm{Sch}_{\mathrm{fppf}}$. Let $p : \mathcal{X} \to (\mathrm{Sch}/S)_{\mathrm{fppf}}$ be a category fibered in groupoids over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. $\mathcal{X}$ is **representable by an algebraic space** if there exists an algebraic space F over S and an equivalence $j : \mathcal{X} \to \mathcal{S}_F$ of categories over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ (example 2.3.8)

Definition 4.4.3. [Sta24, 026N] Let S be a scheme contained in $\mathrm{Sch}_{\mathrm{fppf}}$. An **algebraic stack** over S is a category

$$p : \mathcal{X} \longrightarrow (\mathrm{Sch}/S)_{\mathrm{fppf}}$$

over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ with the following properties

(1) The category $\mathcal{X}$ is a stack in groupoids over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$ (definition 2.3.10).

(2) The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.

(3) There exists a scheme $U \in \mathrm{Ob}((\mathrm{Sch}/S)_{\mathrm{fppf}})$ and a functor $(\mathrm{Sch}/U)_{\mathrm{fppf}} \to \mathcal{X}$ which is surjective and smooth.

Definition 4.4.4. [Sta24, 03YO] Let S be a scheme contained in $\mathrm{Sch}_{\mathrm{fppf}}$. Let $\mathcal{X}$ be an algebraic stack over S. $\mathcal{X}$ is a **Deligne-Mumford stack** if there exists a scheme U and a surjective étale morphism $(\mathrm{Sch}/U)_{\mathrm{fppf}} \to \mathcal{X}$.

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