ALGEBRA QUALIFYING EXAM, FALL 2022

All problems are worth 15 points.

1. Let G be a simple group of order $168 = (2^3)(3)(7)$. For each n below, calculate the number of subgroups of order n inside G. (For some n the answer could be 0).

(a) $n = 7$.

(b) $n = 21$.

(c) $n = 42$.

2. Let p and q be distinct primes. For any $r \geq 1$ consider the group G with presentation

$$\langle a, b | a^p = 1, b^q = 1, bab^{-1} = a^r \rangle.$$

For which r is G a group of order pq? Justify your answer.

3. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, i)$ as a subfield of $\mathbb{C}$, where $i = \sqrt{-1}$ as usual.

(a) Find, with proof, $\text{Gal}(K/Q)$.

(b) Find an element $\beta \in K$ such that $K = \mathbb{Q}(\beta)$.

4. Let K be the splitting field over $\mathbb{Q}$ of an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ of degree 3. Suppose that f has exactly one real root. Prove that $\text{Gal}(K/Q)$ is isomorphic to the symmetric group S_3.

5. Let I be an ideal of a commutative ring R. Suppose that R/I is a flat R-module. Show that $I \cap J = IJ$ for all ideals J of R.

6. Let A be a subring of an integral domain B and let C be the integral closure of A inside of B. Let f and g be monic polynomials with coefficients in B such that all of the coefficients of fg lie in C. Prove that the coefficients of f and g belong to C.