Math 281AB Qualifying Exam

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60/90 Points.

Let $\mathcal{G}$ be a convex and symmetric class of functions $g : \mathbb{R} \to \mathbb{R}$ equipped with norm $\|\cdot\|_{\mathcal{G}}$. Consider a class of functions over $\mathbb{R}$ as follows

$$\mathcal{F} = \left\{ f : \mathbb{R} \to \mathbb{R} | f = \sum_{j=1}^{d} \beta_j \phi_j \text{ where } \sum_{j=1}^{d} \beta_j^2 \leq 1 \right\}.$$

In the above $\{\phi_j\}_{j=1}^{\infty}$ is a Fourier basis.

Suppose we have n, i.i.d. samples of the form

$$y_i = f^*(x_i) + \sigma \varepsilon_i,$$

where each $x_i = (x_{i1}, \cdots, x_{id}) \in \mathbb{R}^d$, ε_i are mean zero Normal random variables with variance 1. Suppose f^* is 2-times differentiable function that belongs to a ball of radius 1: $\int_0^1 [\{f^*\}''(x)]^2 \leq 1$. We estimate f^* by constrained least-squares estimate

$$\hat{f} = \arg\min_{f \in \mathcal{F}} \left\{ n^{-1} \sum_{i=1}^{n} (y_i - f(x_i))^2 \right\}.$$

Prove that the above estimate can be obtained by solving the ridge regression problem, for a suitable choice of regularization parameter.

Prove that

$$E[\|\hat{f} - f\|_n^2] \leq C \left(\frac{\sigma^2}{n} \right)^{4/5}$$

where $C > 0$ is a constant independent of d and n.

NOTE: All details of the computation must be present. Examples and Exercises from the book cannot be used as statements. Theorems and Propositions/Lemmas can be used as statements when and if bounding all the needed terms. Provide ALL the details of your work. Write legibly: points will be taken off if it is impossible to read what was written. Submit your work by sending an email to **jbradic@ucsd.edu**.

Math 281C Qualifying Exam – FALL 2022

(30/100 POINTS) Let $X_1, \ldots, X_n$ be i.i.d. from $N(\mu, \sigma^2)$.

(a) Suppose that $\sigma^2 = \gamma\mu^2$ with unknown $\gamma > 0$ and $\mu \in \mathbb{R}$. Find an LR test for testing $H_0 : \gamma = 1$ versus $H_1 : \gamma \neq 1$.

(b) In the testing problem in (a), find the forms of W_n for Wald's test and R_n for Rao's score test. What can you say about the asymptotic distributions of these two statistics?

(c) Repeat (a) and (b) when $\sigma^2 = \gamma\mu$ with unknown $\gamma > 0$ and $\mu > 0$.

GUIDE: Show all the details in your answers, including the calculations of maximum likelihood estimators and Fisher information matrices (if needed).