Numerical Analysis Qualifying Exam
Fall 2022

September 7, 2022

Instructions:

- There are 8 problems, worth a total of 200 points;
- You must work by yourself on these problems and without help from books or notes or calculators or computers.
- Show the details of your work, such as your scratch work, on each problem to receive credit.

Problems:

1. (25 pts) Suppose $A \in \mathbf{R}^{n\times n}$, for $n \geq 2$, is strictly column diagonally dominant by columns, meaning:

$$|a_{jj}| > \sum_{i=1, i\neq j}^{n} |a_{ij}|,$$

for all $1 \leq j \leq n$. Suppose one step of Gaussian elimination without pivoting is performed on A to arrive at the matrix B (which satisfies $b_{i1} = 0$, for all $2 \leq i \leq n$). Prove $B(2:n, 2:n)$ is also strictly diagonally dominant by columns.

2. (25 pts) Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbf{R}^{2\times 2},$$

where $a_{11}, a_{22} \neq 0$, and $b \in \mathbf{R}^2$. Consider the following iterative method for solving $Ax = b$ with a sequence of approximations, $x^{(k)}$, $k \geq 0$, when given an initial guess $x^{(0)}$: let $x^{(k+1)}$, for each $k \geq 0$, be iteratively generated by the two substeps

- One step of Gauss-Seidel iterations applied to $x^{(k)}$, giving x^*:

  $$x^* = (D - E)^{-1}(Fx^{(k)} + b),$$

  where $A = D - E - F$, for D diagonal, E strictly lower triangular, and F strictly upper triangular;
- One step of Richardson iterations, with parameter ω, applied to x^*, giving $x^{(k+1)}$:

  $$x^{(k+1)} = (I - \omega A)x^* + \omega b.$$

Find conditions on the entries of A and on $\omega \in \mathbf{R}$ that are both necessary and sufficient for the iterative method to converge.

3. (25 pts) For $n \geq 1$, fix $A \in \mathbf{R}^{n\times n}$, a nonsingular matrix, and fix $Q \in \mathbf{R}^{n\times n}$ orthogonal and $R \in \mathbf{R}^{n\times n}$ upper triangular such that $A = RQ$.

Find all combinations of $B, C \in \mathbf{R}^{n\times n}$ such that $\tilde{Q} = BQ$ is orthogonal, $\tilde{R} = RC$ is upper triangular, and $A = \tilde{R}\tilde{Q}$.

4. (25 pts) Suppose $a < b$ and $f \in C^2([a, b])$. Furthermore, suppose b is a root of f and $f'(x) < 0$, $f''(x) > 0$ for all $x \in [a, b]$. Prove Newton's method, applied to f and for all initial guesses in $[a, b]$, will produce a sequence of approximations that converges to b.

5. (25 pts) Remember, the Chebyshev polynomials are $T_n(x)$, for $n \geq 0$, satisfying:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),$$

with $T_0(x) = 1$ and $T_1(x) = x$.

Consider the polynomial

$$p(x) = 3x^3 + 3x^2 - 2x + 4.$$

Among polynomials q of degree 2, find the one that minimizes:

$$\max_{x\in[-1,1]} |p(x) - q(x)|.$$

6. (25 pts) Given $a < b$ and $f \in C^\infty([2a - b, b])$, suppose we are interested in approximating the value of:

$$\int_a^b f(x)\, dx.$$

Given the equally-spaced nodes with stepsize h,

$$x_{-1} < x_0 < x_1 < \cdots < x_n,$$

for $n \geq 1$, with $x_0 = a$ and $x_n = b$, consider the approximation

$$\int_a^b p_n(x)\, dx,$$

where, in the interval $[x_i, x_{i+1}]$, for each $0 \leq i \leq n - 1$, $p_n(x)$ is the quadratic interpolating polynomial for the data points:

$$(x_{i-1}, f(x_{i-1})), (x_i, f(x_i)), (x_{i+1}, f(x_{i+1})).$$

Find $\alpha \in \mathbf{R}$ and $j, k \in \mathbf{Z}$ such that

$$\int_a^b f(x)\, dx - \int_a^b p_n(x)\, dx = \alpha h^j f^{(k)}(\xi),$$

for some $\xi \in [2a - b, b]$.

7. (25 pts) Fix $h > 0$ and consider the approximation of the integral of a function g given by

$$\int_{t+\alpha h}^{t+\beta h} g(\tau)\, d\tau \approx \sum_{j=0}^{s} c_j g(t + \gamma_j h),$$

where:

- For all $0 \le j \le s$, $c_j \in \mathbf{R}$ and $c_j \neq 0$;
- $\alpha, \beta \in \mathbf{Z}$ and, for all $0 \le j \le s$, $\gamma_j \in \mathbf{Z}$;
- $\alpha < \beta$ and, for all $0 \le j < \ell \le s$, $\gamma_j < \gamma_\ell \le \beta$.

Explain how you can use this approximation to derive a difference equation for a linear multistep method with equal-stepsize h for the ODE $y' = f(t, y)$. Additionally:

- Write down the resulting difference equation;
- Determine how many starting values need to be given to use it;
- Determine when it is implicit.

8. (25 pts) Consider the initial value problem with:

- ODE:
  $$y' = f(t, y),$$
  for $t \in [t_0, T]$ with $t_0 < T$, where f is continuous in
  $$D = \{(t, y) \mid t \in [t_0, T], y \in (-\infty, \infty)\},$$
  and Lipschitz continuous in variable y in D;
- Initial value $y(t_0) = y_0$.

Now consider methods applied to this problem of the form

$$y_{i+1} = \alpha y_i + \beta y_{i-1} + h(1 + \beta) f_i,$$

with equal stepsizes $h > 0$ and the additional given exact initial value of $y_1 = y(t_1)$. Find all $\alpha, \beta \in \mathbf{R}$ such that the method is convergent, and plot these as points (α, β) in a graph.