Complex Analysis Qualifying Exam – Fall 2022

Name: ____________________________________

Student ID: ________________________________

Instructions: 3 hours. Open book: Conway and personal notes from lectures may be used. You may use without proof results proved in Conway I-VIII, X-XI. When using a result from the text, be sure to explicitly verify all hypotheses in it. Present your solutions clearly, with appropriate detail.

Notation and terminology: A region is an open and connected subset of $\mathbb{C}$. The space of analytic functions in G is denoted by $H(G)$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
Total		60

Problem 1. [*10 points.*]

Let $G \subset \mathbb{C}$ be a bounded, simply connected region and let f be an analytic self-map of G (i.e., $f(G) \subset G$). Assume that f has two fixed points. Show that $f(z) = z$.

Problem 2. [*10 points.*]

Let $t < 1$. Show that $f(z) = e^{tz} + z + 1$ has exactly one zero (counting multiplicities) in the left halfplane $\{z\colon \operatorname{Re} z < 0\}$.

Hint: Consider first $\operatorname{Re} z < \varepsilon$ for $\varepsilon > 0$.

Problem 3. [*10 points.*]

Show that the punctured unit disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ and the annulus $A = \{z \colon 1 < |z| < 2\}$ are not conformally equivalent.

Problem 4. [*10 points; 4, 6.*]

Let $G \subset \mathbb{C}$ be a bounded region.

(i) Show that if $f \in H(G)$ and $B = B(a, r) \subset G$, then

$$|f(a)| \leq \frac{1}{|B|} \int_B |f(z)| dxdy,$$

where $|B|$ denotes the area of B.

(ii) Show that the space

$$A^1(G) = \{f \in H(G) \colon \|f\|_1 := \int_G |f(z)| dx dy < \infty\}$$

endowed with the metric $d(f, g) = \|f - g\|_1$ is complete.

Note: You may use the result in part (i) even if you did not prove this.

Problem 5. [*10 points; 4, 6.*]

Let $G = \mathbb{C} \setminus \{0\}$, and let $h : G \to \mathbb{R}$ be harmonic.

(i) Show that $g = h_x - ih_y$ is a holomorphic function in G, and that its residue at 0 is a real number.

(ii) Show that there exist a constant c and a holomorphic function $f : G \to \mathbb{C}$ such that

$$h(z) = c \log |z| + \operatorname{Re} f(z) \quad \forall z \in G.$$

Problem 6. [*10 points.*]

If $p \neq 0$ is a polynomial and $a \neq 0$ is a complex number, show that $p(z) - e^{az}$ has infinitely many zeros.